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Negative Sobolev Spaces in the Cauchy Problem for the Cauchy-Riemann Operator

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Let D be a bounded domain in $\mathbb{C}^n$ $(n \geqslant 1)$ with a smooth boundary ∂D. We indicate appropriate Sobolev spaces of negative smoothness to study the non-homogeneous Cauchy problem for the Cauchy-Riemann operator $\overline{\partial}$ in D. In particular, we describe traces of the corresponding Sobolev functions on ∂D and give an adequate formulation of the problem. Then we prove the uniqueness theorem for the problem, describe its necessary and sufficient solvability conditions and produce a formula for its exact solution.

Keywords: negative Sobolev spaces, ill-posed Cauchy problem

As it was understood in the 50-th of XX-th century, it is very natural to consider generalized formulations of boundary value problems and to solve the problems in spaces of generalized functions (see, for instance, [1], [2]). There are two principal reasons for this: the advantage of using very powerful mathematical apparatus of functional analysis and the needs of applications (in modern models it is practically impossible to point-wisely measure the data of boundary value problems; in the best case one can interprete them as functionals). In the present paper we want to consder the Cauchy problem for the Cauchy-Riemann operator $\overline{\partial}$ in spaces of distributions with restrictions on the growth near the boundaries of domains (the last condition is imposed in order to define traces of such distributions on domain's boundaries, see, for instance, [3], [4], [5]).

It is well-known that the Cauchy problem is ill-posed (see, for instance, [6], [7]). However it naturally appears in applications: in Hydrodynamics, in Tomography, in Theory of Electronic Signals. Beginning from the pioneer work [8], the problem was actively studied through the XX-th century (see [7] and [4] for a rather complete bibliography). Here we present the approach developed in [9] for the Cauchy problem for holomorphic functions (cf. [10]); but we consider the *non-homogeneous* Cauchy problem (cf. [11]). Of course, it is easy to see that these problems are equivalent for $n = 1$. On the other hand, if $n > 1$ then the Cauchy-Riemann system is overdetermined, and the equivalence takes place only if we have information on the solvability of the $\overline{\partial}$-equation in a domain where we look for a solution to the problem. Therefore, the problems are not equivalent in domains which have no convexity properties (see, for example, [12], [13]). We emphasize that in the present paper we impose no convexity conditions on the domain D.

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1. Functional Spaces

Let $\mathbb{R}^n$ be n-dimensional Eucledian space and $\mathbb{C}^n$ be n-dimensional complex space with points being n-vectors $z = (z_1, ..., z_n)$, where $z_j = x_j + \sqrt{-1}x_{j+n}$, $j = 1, ..., n$, $\sqrt{-1}$ being imaginary unit and $x = (x_1, ..., x_{2n}) \in \mathbb{R}^{2n}$. We tacitly assume $n > 1$, though $n = 1$ is formally possible.

Let $\overline{\partial}$ be the Cauchy-Riemann operator in $\mathbb{C}^n$. It is known (see, for instance, [13] or [14]) that it induces the differential compatibility complex (Dolbeault complex):

$$0 \longrightarrow \Lambda^{(0,0)} \xrightarrow{\overline{\partial}} \Lambda^{(0,1)} \xrightarrow{\overline{\partial}} \Lambda^{(0,2)} \xrightarrow{\overline{\partial}} ... \xrightarrow{\overline{\partial}} \Lambda^{(0,n)} \longrightarrow 0;$$

here $\Lambda^{(q,r)}$ is the set of all the complex exterior forms of bi-degree (q, r) and $\overline{\partial}$ be the (graduated) Cauchy-Riemann operator extended to the differential forms.

Let D be a bounded domain (i.e., an open connected set) in $\mathbb{R}^{2n}$, and let $\overline{D}$ be its closure. We always assume that the boundary ∂D of D is of class C^∞.

As usual we denote by $C^\infty(D)$ the Frechet space of infinitely differentiable functions in D and by $C^\infty_{comp}(D)$ the space of smooth functions with compact supports in D. Besides, let $C^\infty(\overline{D})$ stand for the set of smooth functions in D with any derivative extending continuously to $\overline{D}$ and, for an open (in the toplology of ∂D) subset $\Gamma \subset \partial D$, let $C^\infty_{comp}(D \cup \Gamma)$ be the set of all $C^\infty(\overline{D})$-functions with compact supports in $D \cup \Gamma$. Everywhere below the set of differential forms of bi-degree (q, r) with the coefficients from a space $\mathfrak{S}(D)$ is denoted by $\mathfrak{S}(D, \Lambda^{(q,r)})$.

Recall that for every $R \in C^\infty(\overline{D}, \Lambda^{(n,n-1)})$ there is such a function $r \in C^\infty(\partial D)$ that $R = rds$ on ∂D (ds being the volume form on ∂D induced from $\mathbb{C}^n$). We will write $R\neg ds$ for r.

Let us denote by $*$ the Hodge operator for differentail forms (see, for instance, [14, §14]); it is convenient to set $\overline{*}f = \overline{*f}$ for $f \in C^\infty(\overline{D}, \Lambda^{(q,r)})$. If $f \in C^\infty(\overline{D}, \Lambda^{(0,1)})$ then $n(f) = \overline{(\overline{*}f)\neg ds}$ is called *normal part* of f on ∂D and then $n(\overline{\partial}v) = \overline{\partial}_n v$ is called the complex normal derivative of a function $v \in C^\infty(\overline{D})$ on ∂D.

We write $L^2(D)$ for the Hilbert space of all the measurable functions in D with a finite norm

$$(u, v)_{L^2(D)} = \int_D u(z)\overline{v}(z)\frac{d\overline{z} \wedge dz}{(2\sqrt{-1})^n}.$$

Then the Hermitian form

$$(u, v)_{L^2(D,\Lambda^{(q,r)})} = \int_D u \wedge \overline{*}v$$

defines the Hilbert structure on $L^2(D, \Lambda^{(q,r)})$.

We also denote by $H^s(D)$ the Sobolev space of distributions over D, whose weak derivatives up to the order $s \in \mathbb{N}$ belong to $L^2(D)$. For a non-integer positive $s \in \mathbb{R}_+$ we define Sobolev spaces H^s with the use of the standard interpolation procedure (see, for example, [15] or [4, §1.4.11]). It is known (see, for instance, [15]) that functions of $H^s(D)$, $s \in \mathbb{N}$, have traces on ∂D of class $H^{s-1/2}(\partial D)$ and the corresponding trace operator is continuous.

Sobolev spaces of negative smoothness may be defined in many different ways (see, for instance, [16]). The standard Sobolev space $H^{-s}(D)$, $s \in \mathbb{N}$, is the completion of $C^\infty(\overline{D})$ with respect to the norm

$$\|u\|_{H^{-s}(D)} = \sup_{v \in C^\infty_{comp}(D)} \frac{|(u, v)_{L^2(D)}|}{\|v\|_{H^s(D)}}.$$

However we prefer to use the ones allowing us to consider boundary value problems, to use integral representations and to get boundedness of standard potentials. That is why we follow [3] (cf. [5], [17]). More precisely, let $C^{\infty,0}(\overline{D})$ be the subspace of $C^\infty(\overline{D})$ having zero values on ∂D. Apart from the standard one, two more types of negative norms may be defined for functions from $C^\infty(\overline{D})$:

$$\|u\|_{-s} = \sup_{v\in C^\infty(\overline{D})} \frac{|(u,v)_{L^2(D)}|}{\|v\|_{H^s(D)}}, \quad |u|_{-s} = \sup_{v\in C^{\infty,0}(\overline{D})} \frac{|(u,v)_{L^2(D)}|}{\|v\|_{H^s(D)}}.$$

It is more correct to write $\|\cdot\|_{-s,D}$ and $|\cdot|_{-s,D}$ but we prefer to drop the index D if this does not cause misunderstandings. Denote the completions of $C^\infty(\overline{D})$ with respect to these norms by $H(D,\|\cdot\|_{-s})$ and $H(D,|\cdot|_{-s})$ respectively. By definition, it is aslo natural to call these Banach spaces *negative Sobolev spaces* (cf. [3]). Obviously, $H(D,\|\cdot\|_{-s}) \hookrightarrow H(D,|\cdot|_{-s}) \hookrightarrow H^{-s}(D)$, because $\|u\|_{-s} \geqslant |u|_{-s} \geqslant \|u\|_{H^{-s}(D)}$.

For $s \in \mathbb{N}$ the Banach space $H(D,\|\cdot\|_{-s})$ is (topologically) isomorphic to the dual space $(H^s(D))'$ for the Hilibert space $H^s(D)$ (see, for instance, [4, Theorem 1.4.28]). This allows us to define a Hilbert structure on the space $H(D,\|\cdot\|_{-s})$. Indeed, this Banach space is a Hilbert one with the scalar product

$$(u,v)_{-s} = \frac{1}{4}\left(\|u+v\|^2_{-s} - \|u-v\|^2_{-s} + \|iu+v\|^2_{-s} - \|iu-v\|^2_{-s}\right),$$

related to the norm $\|.\|_{-s}$, because the mentioned above [4, Theorem 1.4.28]) implies the parallelogramm identity.

Given a bounded domain $\Omega \supset D$, it is easy to see that any $u \in H(D,\|\cdot\|_{-s})$ extends to an element $U \in H(\Omega,\|\cdot\|_{-s})$; for instance, one can set

$$\langle U,v\rangle_\Omega = \langle u,v\rangle_D \text{ for all } v \in H^s(\Omega)$$

(here $\langle .,.\rangle_D$ is the pairing between H and H' for a space H of distributions over D). It is natural to denote this extention by $\chi_D u$; obviously, the support of the distribution $\chi_D u$ lies in $\overline{D}$. The linear operator

$$\chi_D : H(D,\|\cdot\|_{-s}) \to H(\Omega,\|\cdot\|_{-s}),$$

defined in this way is continuous.

In [3] these spaces were used to study Dirichlet problem for the scalar elliptic partial differential operators in $\mathbb{R}^n$ (see also [17] for more general operators); we briefly expose the corresponding results and slightly modify the results [3] for $\mathbb{C}^n$ because we will need them now. To this end, define pairing (u,v) for $u \in H(D,\|\cdot\|_{-s})$, $v \in C^\infty(\overline{D})$ as follows. By the definition, one can find such a sequence $\{u_\nu\}$ in $C^\infty(\overline{D})$ that $\|u_\nu - u\|_{-s} \to 0$ if $\nu \to \infty$. Then

$$|(u_\nu - u_\mu, v)_{L^2(D)}| \leqslant \|u_\nu - u_\mu\|_{-s}\|v\|_{H^s(D)} \to 0 \text{ as } \mu,\nu \to \infty.$$

Set $(u,v) = \lim\limits_{\nu\to\infty}(u_\nu,v)_{L^2(D)}$. It is clear that the limit does not depend on the choice of the sequence $\{u_\nu\}$, for if $\|u_\nu\|_{-s} \to 0$, $\nu \to \infty$, then

$$|(u_\nu,v)_{L^2(D)}| \leqslant \|u_\nu\|_{-s}\|v\|_{H^s(D)}$$

tends to zero, too. This implies that for $u \in H(D,\|\cdot\|_{-s})$ and $v \in C^\infty(\overline{D})$ we have the inequality: $|(u,v)| \leqslant \|u\|_{-s}\|v\|_{H^s(D)}$. Similarly, one defines pairing (u,v) for $u \in H(D,|\cdot|_{-s})$ and $v \in C^{\infty,0}(\overline{D})$ and, obviously one has $|(u,v)| \leqslant |u|_{-s}\|v\|_{H^s(D)}$. Of course, the scalar product $(.,.)_{L^2(D,\Lambda^{(0,1)})}$ induces pairings $(.,.)$ on $H(D,\Lambda^{(0,1)},\|\cdot\|_{-s}) \times C^\infty(\overline{D},\Lambda^{(0,1)})$ and $H(D,\Lambda^{(0,1)},|\cdot|_{-s}) \times C^{\infty,0}(\overline{D},\Lambda^{(0,1)})$.

Now, given F and u_0, consider Dirichlet problem for the Laplace operator $\Delta = \sum\limits_{j=0}^{2n} \frac{\partial^2}{\partial x_j^2}$ in $\mathbb{R}^{2n}$:

$$\left\{ \begin{array}{rcl} \Delta u & = & F \quad \text{in} \quad D, \\ u & = & u_0 \quad \text{on} \quad \partial D. \end{array} \right. \tag{1}$$

More exactly, let $F \in H(D, |\cdot|_{-s-2})$, $u_0 \in H^{-s-1/2}(\partial D)$, $s \in \mathbb{Z}_+$. One says that $u \in H(D, \|\cdot\|_{-s})$ is a *strong solution* to (1) if there is a sequence $\{u_\nu\} \in C^{\infty,0}(\overline{D})$ such that

$$\|u_\nu - u\|_{-s} \to 0,\ \|u_\nu - u_0\|_{-s-1/2,\partial D} \to 0,\ \left\|\overline{\partial}_n u_\nu - \tilde{u}_0\right\|_{-s-3/2,\partial D} \to 0,\ |\Delta u_\nu - F|_{-s-2} \to 0,$$

$\nu \to \infty$, where $\tilde{u}_0 \in H^{-s-3/2}(\partial D)$ is arbitrary.

Given $F \in H(D, |\cdot|_{-s-2})$, $u_0 \in H^{-s-1/2}(\partial D)$, we say that a function u is a *weak solution* to (1) if it belongs to $H(D, \|\cdot\|_{-s'})$ with a number $s' \in \mathbb{Z}_+$ and, according to the Green formula in complex form,

$$(u, \Delta v) = (F, v) - 2(u_0, \overline{\partial}_n v)_{\partial D} \text{ for all } v \in C^{\infty,0}(\overline{D}).$$

Clearly, any strong solution to (1) is a weak one.

Theorem 1. *Let $s \in \mathbb{Z}_+$. If $F \in H(D, |\cdot|_{-s-2})$, $u_0 \in H^{-s-1/2}(\partial D)$, then there is the unique weak solution u to the problem* (1)*. In particular, the weak solution to* (1) *is the strong one and*

$$\|u\|_{-s} \leqslant c\ \left(|F|_{-s-2} + \|u_0\|_{-s-1/2,\partial D}\right),$$

where the constant c does not depend on F, u_0 and u.

Proof. See, for instance, [17] (cf. [3] for the real case). □

Denote by $\mathcal{P}^{(D)} : H^{-s-1/2}(\partial D) \to H(D, \|\cdot\|_{-s})$ the continuous operator, mapping u_0 and $F = 0$ to the unique solution to Dirichlet problem (1). Of course, on a sufficiently smooth u_0, this is nothing but the Poisson integral of the Dirichlet problem. Similarly, denote by $\mathcal{G}^{(D)}$: $H(D, |\cdot|_{-s-2}) \to H(D, \|\cdot\|_{-s})$ the continuous operator, mapping $F \in H(D, |\cdot|_{-s-2})$ to the unique solution to Dirichlet problem (1) with the zero boundary data.

Now we want to solve the Cauchy problem for $\overline{\partial}$ in spaces $H(D, \|\cdot\|_{-s})$. For an element u of $H^s(D)$, $H(D, \|\cdot\|_{-s})$ or $H(D, |\cdot|_{-s})$ we always understand $\overline{\partial} u$ in the sense of distributions in D. Of course, $\overline{\partial}$ continuously maps $H^s(D)$ to $H^{s-1}(D)$, $s \in \mathbb{Z}$.

Lemma 1. *The differential operator $\overline{\partial}$ induces a linear bounded operator*

$$\overline{\partial} : H(D, \Lambda^{(0,r)}, \|\cdot\|_{-s}) \to H(D, \Lambda^{(0,r+1)}, |\cdot|_{-s-1}).$$

Proof. It immediately follows from Stokes' formula. □

However there is no need for elements of $H(D, \|\cdot\|_{-s})$ to have traces on ∂D and there is no need for $\overline{\partial}$ to map $H(D, \|\cdot\|_{-s})$ to $H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$.

For this reason we introduce two more types of spaces (cf. [4, §9.2, 9.3]). Namely, denote the completion of $C^\infty(\overline{D})$ with respect to the graph norms

$$\|u\|_{-s,\overline{\partial}} = \left(\|u\|^2_{-s} + \|\overline{\partial} u\|^2_{-s-1}\right)^{1/2}, \quad \|u\|_{-s,b} = \left(\|u\|^2_{-s} + \|u\|^2_{-s-1/2,\partial D}\right)^{1/2}$$

by $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ and $H_b(D, \|\cdot\|_{-s})$ respectively. Obviously, $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ and $H_b(D, \|\cdot\|_{-s})$ are Hilbert spaces with scalar products

$$(u, v)_{-s,\overline{\partial}} = (u, v)_{-s} + (\overline{\partial} u, \overline{\partial} v)_{-s-1}, \quad (u, v)_{-s,b} = (u, v)_{-s} + (u, v)_{-s-1/2,\partial D}$$

respectively. Clearly, elements of these spaces are more regular in $\overline{D}$ than elements of $H(D, \|\cdot\|_{-s})$. Moreover, by the definition, the differential operator $\overline{\partial}$ induces a bounded linear operator

$$\overline{\partial}_{-s} : H_{\overline{\partial}}(D, \|\cdot\|_{-s}) \to H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1}),$$

and the trace operator $t_s : H^s(D) \to H^{s-1/2}(\partial D)$ induces a bounded linear trace operator

$$t_{-s} : H_b(D, \|\cdot\|_{-s}) \to H^{-s-1/2}(\partial D).$$

Theorem 2. *The linear spaces $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ and $H_b(D, \|\cdot\|_{-s})$ coincide and their norms are equivalent.*

Proof. By the definitions of the spaces we need to check the equivalence of norms on $C^\infty(\overline{D})$ only. Let $\overline{\partial}^* g = -\overline{*}\overline{\partial}\overline{*}g$ be the formal adjoint for $\overline{\partial}$. Then because of Stokes' formula we have:

$$(\overline{\partial}v, g) = (v, \overline{\partial}^* g) + \int_{\partial D} v(\overline{*}g) \text{ for all } g \in C^\infty(\overline{D}, \Lambda^{(0,1)}),\ v \in C^\infty(\overline{D}).$$

Hence, for all $v \in C^\infty(\overline{D})$ we have:

$$\|v\|^2_{-s} + \|\overline{\partial}v\|^2_{-s-1} \leqslant (\|\overline{\partial}^*_{s+1}\|^2 + \|n_{s+1}\|^2 + 1)(\|v\|^2_{-s} + \|v\|^2_{-s-1/2,\partial D}),$$

where n_{s+1} is the continuous operator $n_{s+1} : H^{s+1}(D, \Lambda^{(0,1)}) \to H^{s+1/2}(\partial D)$ induced by the normal operator n, and $\overline{\partial}^*_{s+1} : H^{s+1}(D, \Lambda^{(0,1)}) \to H^s(D)$ is the continuous operator induced by $\overline{\partial}^*$.

Back, fix a defining function $\rho \in C^\infty$ of the domain D; without loss of a generality we may assume $|d\rho| = 1$ on ∂D. For a function $g_0 \in C^\infty(\partial D)$, set

$$G_0 = \sum_{j=1}^n \mathcal{P}^{(D)} \left(\frac{g_0 \frac{\partial \rho}{\partial \overline{z}_j}}{\left(\sum\limits_{k=1}^n \left| \frac{\partial \rho}{\partial \overline{z}_k} \right|^2 \right)^{1/2}} \right) d\overline{z}_j.$$

Due to [14, lemma 3.5] and the properties of the Poisson integral $\mathcal{P}^{(D)}$, we see that $G_0 \in C^\infty(\overline{D}, \Lambda^{(0,1)}))$ with $n(G_0) = g_0$ on ∂D and

$$\|G_0\|_{H^{s+1}(D,\Lambda^{(0,1)})} \leqslant \gamma \|g_0\|_{H^{s+1/2}(\partial D)}$$

with a constant $\gamma = \gamma(s)$, not depending on g_0 and G_0. Then, by Stokes' formula, we have:

$$\int_{\partial D} v g_0 ds(x) = \int_{\partial D} v(\overline{*}G_0) = (\overline{\partial}v, G_0) - (v, \overline{\partial}^* G_0) \text{ for all } v \in C^\infty(\overline{D}).$$

Hence

$$\|v\|^2_{-s} + \|v\|^2_{-s-1/2} \leqslant (1 + \gamma^2 + \gamma^2 \|\overline{\partial}^*_{s+1}\|^2)(\|v\|^2_{-s} + \|\overline{\partial}v\|^2_{-s-1}). \qquad \square$$

2. Weak Boundary Values of Sobolev functions

Consider now the weak extension of $\overline{\partial}$ on the scale $H(D, \|\cdot\|_{-s})$. Namely, denote by $H^w_{\overline{\partial}}(D, \|\cdot\|_{-s})$ the set of functions u from $H(D, \|\cdot\|_{-s})$ such that $\overline{\partial}u \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$. As $\overline{\partial}$ is linear, then the set is linear too; we endow it with the graph norm

$$\|u\|_{-s,\overline{\partial},w} = \left(\|u\|^2_{-s} + \|\overline{\partial}u\|^2_{-s-1}\right)^{1/2}.$$

It is not difficult to see that the normed space $H^w_{\overline{\partial}}(D, \|\cdot\|_{-s})$ is complete.

Clearly,

$$H_{\overline{\partial}}(D, \|\cdot\|_{-s}) \subset H^w_{\overline{\partial}}(D, \|\cdot\|_{-s}), \tag{2}$$

Besides, the differential operator $\overline{\partial}$ induces the linear bounded operator

$$\overline{\partial}^w_{-s} : H^w_{\overline{\partial}}(D, \|\cdot\|_{-s}) \to H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1}).$$

The unions $\cup_{s=1}^\infty H^w_{\overline{\partial}}(D, \|\cdot\|_{-s})$ and $\cup_{s=1}^\infty H(D, \|\cdot\|_{-s})$ we denote by $H_{\overline{\partial}}(D)$ and $H(D)$ respectively.

As before, let Γ be an open (in the topology of ∂D) connected subset of ∂D.

Definition 1. *We say that a function $u \in H_{\overline{\partial}}(D)$ has weak boundary value $t^w_\Gamma(u) = u_0 \in \mathcal{D}'(\Gamma)$ with respect to the operator $\overline{\partial}$ on Γ if*

$$(\overline{\partial} u, g)_D - (u, \overline{\partial}^* g)_D = (u_0, n(g)) \text{ for all } g \in C^\infty_{comp}(D \cup \Gamma, \Lambda^{(0,1)}).$$

Stokes' formula implies that every $u \in H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ has a weak boundary value on ∂D in the sense of Definition 1, coinciding with the trace $t_{-s}(u) \in H^{-s-1/2}(\partial D)$.

Theorem 3. *For every function $u \in H_{\overline{\partial}}(D)$ there is the weak boundary value $t^w_{\partial D}(u)$ in the sense of Definition 1, coinciding with limit boundary value of the harmonic function $(u+\mathcal{G}^{(D)}(2\overline{\partial}^*(\overline{\partial} u)))$ of finite order of growth near ∂D.*

Proof. Let $u \in H_{\overline{\partial}}(D)$. Then there is such an $s \in \mathbb{N}$ that $u \in H^\omega_{\overline{\partial}}(D, \|\cdot\|_{-s})$, and then $\overline{\partial} u \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$.

First of all, we note that by Lemma 1 and Theorem 1 the operator $\mathcal{G}^{(D)}\overline{\partial}^*$ continuously maps the space $H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$ to $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$.

Then any element w of the image $\mathcal{G}^{(D)}\overline{\partial}^*(H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1}))$ has zero trace $t_{-s}(w)$, and hence it has zero weak boundary value on ∂D in the sense of Definition 1. Now it is clear that a function $u \in H_{\overline{\partial}}(D)$ has weak boundary value $t^w_{\partial D}(u)$ in the sense of Definition 1 if and only if the function $v = (u + \mathcal{G}^{(D)}(2\overline{\partial}^*(\overline{\partial} u)))$ does. Since $\Delta = -2\overline{\partial}^*\overline{\partial}$ we see that, by the construction, $v \in H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ satisfies

$$\Delta v = -2\overline{\partial}^*\overline{\partial} u + 2\overline{\partial}^*(\overline{\partial} u) = 0 \text{ in } D.$$

In particular, as $v \in H(D, \|\cdot\|_{-s})$, it has a finite order of growth near ∂D and has weak limit value $t(v) = v_0 \in \mathcal{D}'(\partial D)$ (see [5]). More precisely, set $D_\varepsilon = \{x \in D : \rho(x) < -\varepsilon\}$. Then, for a sufficiently small $\varepsilon > 0$, the sets $D_\varepsilon \Subset D \Subset D_{-\varepsilon}$ are domains with smooth boundaries $\partial D_{\pm\varepsilon}$ of class C^∞ and vectors $\mp\varepsilon\nu(x)$ belong to $\partial D_{\pm\varepsilon}$ for every $x \in \partial D$ (here $\nu(x)$ is the external normal unit vector to the hypersurface ∂D at the point x). It is said that $v = v_0$ in the sense of weak limit values on Γ if

$$< v_0, w >= \lim_{\varepsilon \to +0} \int_{\partial D} w(y) v(y - \varepsilon\nu(y)) ds(y) \text{ for all } w \in C^\infty_{comp}(\Gamma).$$

Further, as it is explained above,

$$\langle \chi_D v, \overline{w} \rangle = (v, w)_D \text{ for all } w \in C^\infty(\mathbb{C}^n),$$

$$\langle \chi_D(\overline{\partial} v), \overline{*G} \rangle = (\overline{\partial} v, G)_D \text{ for all } G \in C^\infty(\mathbb{C}^n, \Lambda^{(0,1)}).$$

By the construction, $(\overline{\partial}\oplus\overline{\partial}^*)\overline{\partial}v = 0$ in D and the components of $\overline{\partial}v$ are harmonic functions with a finite order of growth near ∂D. Ellipticity of the operators Δ and $\overline{\partial}\oplus\overline{\partial}^*$ [4, Theorem 9.4.7] implies that there is a positive sequence $\{\varepsilon_\nu\}$, converging to zero and such that

$$\langle \chi_D v, \overline{w}\rangle = \lim_{\varepsilon_\nu\to+0}\int_{D_{\varepsilon_\nu}} v(x)\overline{w}(x)dx \text{ for all } w\in C^\infty(\mathbb{C}^n),$$

$$\langle \chi_D(\overline{\partial}v), \overline{*}G\rangle = \lim_{\varepsilon_\nu\to+0}\int_{D_{\varepsilon_\nu}} \overline{\partial}v\wedge\overline{*}G \text{ for all } G\in C^\infty(\mathbb{C}^n,\Lambda^{(0,1)}).$$

By the Whitney Theorem, any smooth function on $\overline{D}$ can be extended to a smooth function on $\mathbb{C}^n$. Therefore

$$(v,w)_D = \lim_{\varepsilon_\nu\to+0}\int_{D_{\varepsilon_\nu}} v(x)\overline{w}(x)dx \text{ for all } w\in C^\infty(\overline{D}),$$

$$(\overline{\partial}v, G)_D = \lim_{\varepsilon_\nu\to+0}\int_{D_{\varepsilon_\nu}} \overline{\partial}v\wedge(\overline{*}G) \text{ for } G\in C^\infty(\overline{D},\Lambda^{(0,1)}).$$

Hence, by Stokes' formula, for all $g\in C^\infty(\overline{D},\Lambda^{(0,1)})$ we have:

$$(\overline{\partial}v, g)_D - (v,\overline{\partial}^* g)_D = \lim_{\varepsilon_\nu\to+0}\int_{D_{\varepsilon_\nu}} (\overline{\partial}v\wedge(\overline{*}g) - v\overline{*}\overline{\partial}^* g) =$$

$$\lim_{\varepsilon_\nu\to+0}\int_{\partial D_{\varepsilon_\nu}} v\overline{*}g = \lim_{\varepsilon_\nu\to+0}\int_{\partial D_{\varepsilon_\nu}} v((\overline{*}g)\neg ds_{\varepsilon_\nu})ds_{\varepsilon_\nu} = (v_0, n(g)),$$

which was to be proved. □

Corollary 1. *For every function $u\in H^w_{\overline{\partial}}(D,\|\cdot\|_{-s})$ there is weak limit value $t^w_{\partial D}(u)$ on ∂D in the sense of Definition 1, belonging to $H^{-s-1/2}(\partial D)$, with*

$$\|t^w_{\partial D}(u)\|_{-s-1/2,\partial D}\leqslant C\,\|u\|_{-s,\overline{\partial}}, \tag{3}$$

where the constant C does not depend on u.

Proof. We have already proved the existence of weak boundary values in the sense of Definition 1 in the class of distributions for elements of the space $H^w_{\overline{\partial}}(D,\|\cdot\|_{-s})$. We need to prove that they belong to the corresponding Sobolev spaces on ∂D. Fix $g_0\in C^\infty(\partial D)$. Then, as we have seen proving Theorem 2,

$$|(u_0,g_0)| = |(u_0,n(G_0))| = \left|(\overline{\partial}u, G_0) - (u,\overline{\partial}^* G_0)\right| \leqslant$$

$$\leqslant \|\overline{\partial}u\|_{-s-1}\|G_0\|_{H^{s+1}(D,\Lambda^{(0,1)})} + \|u\|_{-s}\|\overline{\partial}^* G_0\|_{H^s(D,\Lambda^{(0,1)})} \leqslant C\|u\|_{-s,\overline{\partial}}\|g_0\|_{H^{s+1/2}(\partial D)}$$

with a constant C not depending on g_0. Hence

$$\|u_0\|_{H^{-s-1/2}(\partial D)} = \sup_{\phi\in C^\infty_{comp}(\partial D)}\frac{|(u_0,\phi)|}{\|\phi\|_{H^{s+1/2}(\partial D)}}\leqslant C\|u\|_{-s,\overline{\partial}}.$$

Thus, we have proved that $t^w_{\partial D}(u)\in H^{-s-1/2}(\partial D)$ and the estimate (3) holds true. □

Corollary 2. *The spaces $H_{\overline{\partial}}(D,\|\cdot\|_{-s})$ and $H^w_{\overline{\partial}}(D,\|\cdot\|_{-s})$ coincide.*

Proof. Because of (2), it is sufficient to prove that

$$H^w_{\overline{\partial}}(D, \|\cdot\|_{-s}) \subset H_{\overline{\partial}}(D, \|\cdot\|_{-s}).$$

Fix a function $u \in H^w_{\overline{\partial}}(D, \|\cdot\|_{-s})$. By Corollary 1, it has weak boundary value $t^w_{\partial D}(u) \in H^{-s-1/2}(\partial D)$. Let us show that u is a weak solution to the Dirichlet problem (1) with the data

$$F = -2\overline{\partial}^*(\overline{\partial} u) \in H(D, |\cdot|_{-s-2}) \text{ and } t^w_{\partial D}(u) \in H^{-s-1/2}(\partial D).$$

Indeed, Corollary 1 implies that, for all $v \in C^{\infty,0}(\overline{D})$, we have:

$$(u, \Delta v) = -2(u, \overline{\partial}^*(\overline{\partial} v)) = -2(\overline{\partial} u, \overline{\partial} v) + 2(u_0, \overline{\partial}_n v). \tag{4}$$

On the other hand, as $\overline{\partial} u \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$, then there is a sequence $\{f_\nu\} \subset C^\infty(\overline{D}, \Lambda^{(0,1)})$ converging to $\overline{\partial} u$ in this space. By Lemma 1, the sequence $\{\overline{\partial}^* f_\nu\} \subset C^\infty(\overline{D})$ converges to $\overline{\partial}^*(\overline{\partial} u)$ in the space $H(D, |\cdot|_{-s-2})$. That is why

$$(\overline{\partial} u, \overline{\partial} v) = \lim_{\nu\to\infty} (f_\nu, \overline{\partial} v) = \lim_{\nu\to\infty} (\overline{\partial}^* f_\nu, v) = (\overline{\partial}^*(\overline{\partial} u), v). \tag{5}$$

Combining (4) and (5) we conclude that u is the weak solution to the Dirichlet problem.

Finally, Theorem 1 implies that u is the strong solution to the Dirichlet problem and hence it belongs to $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$. □

Corollary 3. *Let $u \in H(D)$. If*

$$\overline{\partial} u \in H(D, \|\cdot\|_{-s-1}), \quad t_{-s}(u) \in H^{-s-1/2}(\partial D)$$

then $u \in H_{\overline{\partial}}(D, \|\cdot\|_{-s})$. Moreover, if $\overline{\partial} u \in H^{s-1}(D)$, $t(u) \in H^{s-1/2}(\partial D)$ then $u \in H^s(D)$.

As we have seen above, the space $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ is a suitable class for stating the Cauchy problem for the Cauchy-Riemann operator. In order to do this we need to choose proper spaces for the boundary Cauchy data on a surface $\Gamma \subset \partial D$. As we are interesting in the case $\Gamma \neq \partial D$, we will use one more type of Sobolev spaces: Sobolev spaces on closed sets (see, for instance, [4, §1.1.3]). Namely, let $H^{-s}(\overline{\Gamma})$ stand for the factor space of $H^{-s}(\partial D)$ over the subspace of functions vanishing on a neighbourhood of $\overline{\Gamma}$. Of course, it is not so easy to handle this space, but its every element extends from Γ up to an element of $H^{-s}(\partial D)$. Further characteristic of this space may be found in [4, Lemma 12.3.2]). We only note that if Γ has a C^∞-smooth boundary (on ∂D), then

$$H(\Gamma, \|\cdot\|_{-s,\Gamma}) \hookrightarrow H^{-s}(\overline{\Gamma}) \hookrightarrow H^{-s}(\Gamma).$$

Corollary 4. *For every function $u \in H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ and every $\Gamma \subset \partial D$ there is boundary value $t_\Gamma(u)$ in the sense of Definition 1, belonging to $H^{-s-1/2}(\overline{\Gamma})$.*

As ∂D is compact, $\cup_{s=1}^\infty H(\partial D, \|\cdot\|_{-s,\partial D}) = \mathcal{D}'(\partial D)$. Set $\cup_{s=1}^\infty H^{-s}(\overline{\Gamma}) = \mathcal{D}'(\overline{\Gamma})$.

Corollary 5. *For every $u \in H_{\overline{\partial}}(D)$ and every $\Gamma \subset \partial D$ there is boundary value $t_\Gamma(u)$ in the sense of Definition 1, belonging to $\mathcal{D}'(\overline{\Gamma})$.*

3. The Martinelli-Bochner Formula

Let Φ be the standard fundamental solution to the Laplace operator in $\mathbb{R}^{2n}$ and $\mathfrak{U}$ be the Martinelli-Bochner kernel (see, for instance, [14]):

$$\Phi(x-y) = \begin{cases} \frac{1}{2\pi}\frac{1}{\ln|x-y|}, & n=1, \\ \frac{1}{\sigma_{2n}}\frac{1}{(2-2n)|x-y|^{2n-2}}, & n>1, \end{cases} \qquad \mathfrak{U}(\zeta,z) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n}\sum_{j=1}^{n}(-1)^{j-1}\frac{\overline{\zeta}_j-\overline{z}_j}{|\zeta-z|^{2n}}d\overline{\zeta}[j]\wedge d\zeta$$

where σ_n is the square of the unit sphere in $\mathbb{R}^n$. We use the same notation Φ also for the operator corresponding to the introduced fundamental solution kernel.

For $z \notin \partial D$, denote by $Mv_0(z)$ the Martinelli-Bochner transform of a density $v_0 \in \mathcal{D}'(\partial D)$, i.e., the action's result of the distribution v_0 to the function $n(\mathfrak{U}(\cdot,z))$ with respect to the variable $\zeta \in \partial D$. As the kernel $\mathfrak{U}$ is harmonic with respect to $z \neq \zeta$, the transform is harmonic everywhere in $\mathbb{C}^n$ outside the support $\operatorname{supp} v_0$ of the density v_0.

Further, for a form $f \in C^\infty(\overline{D}, \Lambda^{(0,1)})$ denote by $T_D f$ the following volume potential:

$$(T_D f)(z) = (\Phi\overline{\partial}^*\chi_D f)(z) = \int_D f(\zeta)\wedge\mathfrak{U}(\zeta,z).$$

Lemma 2. *For every bounded domain $\Omega \Subset \mathbb{C}^n$ with $\partial\Omega \in C^\infty$ and $\partial D \cap \Omega = \emptyset$, the potential T_D induces the bounded linear operator*

$$T_{D,\Omega} : H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1}) \to H_{\overline{\partial}}(\Omega, \|\cdot\|_{-s}).$$

Moreover, for any form $f \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$, the function $T_{D,\Omega} f$ is harmonic in $\Omega \setminus \overline{D}$.

Proof. For all $g \in C^\infty(\overline{D}, \Lambda^{(0,1)})$ and $\phi, \psi \in C^\infty(\overline{\Omega})$ we have:

$$(T_D g, \phi)_\Omega = (\Phi\overline{\partial}^*\chi_D g, \chi_\Omega\phi)_{\mathbb{C}^n} = (\chi_D g, \overline{\partial}\Phi\chi_\Omega\phi)_{\mathbb{C}^n},$$

$$(\overline{\partial}T_D g, \psi)_\Omega = (\overline{\partial}\Phi\overline{\partial}^*\chi_D g, \chi_\Omega\psi)_{\mathbb{C}^n} = (\chi_D g, \overline{\partial}\Phi\overline{\partial}^*\chi_\Omega\psi)_{\mathbb{C}^n} = (\chi_D g, \overline{\partial}T_\Omega\psi)_{\mathbb{C}^n}.$$

As pseudo-differential operators $\Phi\chi_\Omega$ and T_Ω are continuous on the scale of Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{Z}_+$ (see, for instance, [4, theorem 2.4.24] then

$$\|T_D g\|_{-s,\overline{\partial},\Omega} \leqslant C\,\|g\|_{-s,D} \text{ for all } g \in C^\infty(\overline{D}, \Lambda^{(0,1)}), \tag{6}$$

with a constant $C > 0$ does not depending on $g \in C^\infty(\overline{D}, \Lambda^{(0,1)})$.

Now, if $f \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$ then there is a sequence $\{f_\nu\} \subset C^\infty(\overline{D}, \Lambda^{(0,1)})$ converging to f in $H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$. By the inequality (6), the sequence $\{T_D f_\nu\}$ is fundamental in $H_{\overline{\partial}}(\Omega, \|\cdot\|_{-s})$; we its limit denote by $T_{D,\Omega} f$. Clearly this limit does not depend on the choice of the sequence $\{f_\nu\} \subset C^\infty(\overline{D}, \Lambda^{(0,1)})$, and the estimate (6) guarantees the boundedness of the linear operator $T_{D,\Omega} f$ defined in this way. Moreover, since every potential $T_D f_\nu$ is harmonic in $\mathbb{C}^n \setminus \overline{D}$ then Stiltjes-Vitali Theorem implies that the sequence $\{T_D f_\nu\}$ converges uniformly together with all the derivatives on compacts in $\Omega \setminus \overline{D}$ and its limit is harmonic in $\Omega \setminus \overline{D}$. $\square$

Lemma 3. *For any domain $\Omega \Subset \mathbb{C}^n$ such that $\partial\Omega \in C^\infty$ and $D \subset \Omega$, the transform M defined above induces bounded linear operators*

$$M_D : H^{-s-1/2}(\partial D) \to H_{\overline{\partial}}(D, \|\cdot\|_{-s}),$$

$$M_\Omega : H^{-s-1/2}(\partial D) \to H(\Omega, \|\cdot\|_{-s}).$$

Besides, for every function $v \in H_{\overline{\partial}}(D)$ the Martinelli-Bochner formulae hold true:

$$M_D(t(v)) + T_{D,D}\overline{\partial}v = v, \quad M_\Omega(t(v)) + T_{D,\Omega}\overline{\partial}v = \chi_D v. \tag{7}$$

Proof. As we have seen before, for any $v_0 \in H^{-s-1/2}(\partial D)$, the Poisson integral $\mathcal{P}^{(D)}(v_0) \in H_{\overline{\partial}}(\overline{D}, \|\cdot\|_{-s})$, satisfies $t(\mathcal{P}^{(D)}(v_0)) = v_0$ (see Theorem 1). We set

$$M_D = \mathcal{P}^{(D)} - T_{D,D}\overline{\partial}\mathcal{P}^{(D)} : H^{-s-1/2}(\partial D) \to H_{\overline{\partial}}(D, \|\cdot\|_{-s}),$$

$$M_\Omega = \chi_D\mathcal{P}^{(D)} - T_{D,\Omega}\overline{\partial}\mathcal{P}^{(D)} : H^{-s-1/2}(\partial D) \to H(\Omega, \|\cdot\|_{-s}).$$

By Lemma 2, Theorem 1 and the continuity of the operator χ_D, the operators M_D, M_Ω, defined in this way, are continuous. Let us show that M_D and M_Ω coincide with the transform M on $C^\infty(\partial D)$. Indeed, if $v_0 \in C^\infty(\partial D)$ then $\mathcal{P}^{(D)}v_0 \in C^\infty(\overline{D})$ and

$$Mv_0 = M(t(\mathcal{P}^{(D)}v_0)).$$

Then, by Martinelli-Bochner formula for smooth functions (see [14]), we have:

$$\chi_D\mathcal{P}^{(D)}v_0 = Mv_0 + T_D\overline{\partial}\mathcal{P}^{(D)}v_0,$$

which was to be proved.

As $C^\infty(\partial D)$ is dense in $H^{-s-1/2}(\partial D)$ then M continuously extends from $C^\infty(\partial D)$ to the space $H^{-s-1/2}(\partial D)$ up to the operators M_D, M_Ω defined above. Moreover, it is easy to see that the functions $M_D v_0$, $M_\Omega v_0$ coincide with the transform Mv_0 in D and $\Omega \setminus \operatorname{supp} v_0$ respectively. Indeed, let $\phi \in C^\infty_0(\Omega \setminus \operatorname{supp} v_0)$. We approximate the distribution $v_0 \in H^{-s-1/2}(\partial D)$ by smooth functions $v_0^{(\nu)}$ with supports in a neighbourhood of $\operatorname{supp} v_0$ in such a way that $\operatorname{supp} v_0^{(\nu)} \cap \operatorname{supp}\phi = \emptyset$. Then, by easy computations, $\lim_{\nu\to\infty}(M(v_0 - v_0^{(\nu)}), \phi)_\Omega = 0$, and hence

$$(Mv_0, \phi)_\Omega = \lim_{\nu\to\infty}(Mv_0^{(\nu)}, \phi)_\Omega = (M_\Omega v_0, \phi)_\Omega$$

because M_Ω is continuous. Similarly, if $\operatorname{supp}\phi \subset D$ then

$$(Mv_0, \phi)_D = \lim_{\nu\to\infty}(Mv_0^{(\nu)}, \phi)_D = (M_D v_0, \phi)_D.$$

Let now $v \in H_{\overline{\partial}}(D)$. Then $v \in H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ with a number s and there is a sequence $\{v_\nu\} \subset C^\infty(\overline{D})$ converging to v in $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$. Therefore the Martinelli-Bochner formula for smooth functions implies

$$M(t(v_\nu)) + T_D\overline{\partial}v_\nu = \chi_D v_\nu. \tag{8}$$

Passing to the limit with respect to $\nu \to \infty$ in the spaces $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ and $H(\Omega, \|\cdot\|_{-s})$ in (8) we obtain (7) because of Lemma 2 and the continuity of M_D, M_Ω, proved above. □

Remark 1. *Let $f \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$. If Ω, Ω_1 and Ω_2 are bounded domains in $\mathbb{C}^n$ containing D, having smooth boundaries and such that $\Omega_p \subset \Omega$, p=1,2, then functions $T_{D,\Omega}f \in H(\Omega, \|\cdot\|_{-s})$, $T_{D,\Omega_1\setminus\overline{D}}f \in H(\Omega_1 \setminus \overline{D}, \|\cdot\|_{-s})$ and $T_{D,\Omega_2\setminus\overline{D}}f \in H(\Omega_2 \setminus \overline{D}, \|\cdot\|_{-s})$ are harmonic in $\Omega \setminus \overline{D}$, $\Omega_1 \setminus \overline{D}$ and $\Omega_2 \setminus \overline{D}$ respectively. As each of them is constructed as a limit of the same sequence of functions, they coincide in $(\Omega_1 \cap \Omega_2)\setminus\overline{D}$. Actually, as Ω, Ω_1, and Ω_2 are arbitrary, all these limits harmonically extend their to $\mathbb{C}^n \setminus \overline{D}$ and all these extentions coincide, too. Since the operators M_Ω, M_{Ω_1}, and M_{Ω_2} are constructed with the use of the operators $T_{D,\Omega}$ $T_{D,\Omega_1\setminus\overline{D}}$ and $T_{D,\Omega_2\setminus\overline{D}}$ respectively, this remark is valid for potentials of the type $M_\Omega v_0$ with $v_0 \in H^{-s-1/2}(\partial D)$. This allows us to consider functions $T_D f$ and Mv_0 harmonic in $\mathbb{C}^n \setminus \overline{D}$, having finite orders of growth near ∂D (outside $\overline{D}$!) and such that $T_D f = T_{D,\Omega} \in H(\Omega, \|\cdot\|_{-s})$, $Mv_0 \in H(\Omega, \|\cdot\|_{-s})$ for any domain $\Omega \supset D$.*

4. The Cauchy problem

Set $H(D,\Lambda^{(0,1)}) = \cup_{s=1}^{\infty} H(D,\Lambda^{(0,1)},\|\cdot\|_{-s})$.

Problem 1. *Given $f \in H(D,\Lambda^{(0,1)})$, $u_0 \in \mathcal{D}'(\overline{\Gamma})$, find $u \in H_{\overline{\partial}}(D)$ with*

$$(u,\overline{\partial}^*\phi) = (f,\phi) - (u_0,n(\phi)) \text{ for all } \phi \in C^{\infty}_{comp}(D\cup\Gamma,\Lambda^{(0,1)}). \tag{9}$$

As we have seen before, a function $u \in H(D)$ is a solution to the Cauchy Problem 1 if and only if $\overline{\partial} u = f$ in the sense of distributions in D and $t_{\Gamma}(u) = u_0$ on $\overline{\Gamma}$. Moreover, Corollary 3 means that, for sufficiently smooth data f and u_0, Problem 1 becomes the classical Cauchy problem for the Cauchy-Riemann operator. Besides, we easily get the Uniqueness Theorem for Problem 1.

Theorem 4. *Problem 1 has no more than one sulution.*

Proof. Indeed, if $u_0 = 0$, $f = 0$ then corollary 1 implies that a solution to 1 is a holomorphic Sobolev function in D having zero limit values on Γ. As it has a finite order of growth near Γ (see, for instance, [4, theorem 9.4.8]), we conclude that $u \equiv 0$ in D because of [4, theorem 10.3.5]). □

As $\overline{\partial}^2 = 0$, then $\overline{\partial} f = 0$ in D if the Cauchy problem is solvable. Besides, if $n > 1$, the Cauchy-Riemann operator $\overline{\partial}$ induces the tangential operator $\overline{\partial}_{\tau}$ on ∂D (see, for instance, [14, §11]). This means that the Cauchy data u_0 and f have to be coherent. Namely, taking in (9) as ϕ a differential form $\overline{\partial}^*\beta$ with $\beta \in C^{\infty}_{comp}(D\cup\Gamma,\Lambda^{(0,2)})$, we see that

$$(u_0,n(\overline{\partial}^*\beta)) = (f,\overline{\partial}^*\beta) \text{ for all } \beta \in C^{\infty}_{comp}(D\cup\Gamma,\Lambda^{(0,2)}), \tag{10}$$

if Problem 1 is solvable. For $f = 0$ it means that u_0 is a CR-function on Γ.

We want to get a solvability criterion for Problem 1. With this aim, let us choose a domain D^+ in such a way that the set $\Omega = D\cup\Gamma\cup D^+$ would be a bounded domain with a smooth boundary; it is convenient to set $D^- = D$. For a function $v \in C(D^+\cup D^-)$ we denote by $v^{\pm}$ its restriction to $D^{\pm}$.

For $u_0 \in H^{-s-1/2}(\overline{\Gamma})$ we fix an element $\tilde{u}_0 \in H^{-s-1/2}(\partial D)$ of its equivalence class. As we have explained in Remark 1, the distribution $F = M\tilde{u}_0 + T_D f$ is harmonic outside $\overline{D}$ and belongs to $H(\Omega,\|\cdot\|_{-s})$.

Theorem 5. *The Cauchy problem* 1 *is solvable if and only if condition* (10) *holds and there is a harmonic in Ω function $\mathcal{F}$ of finite order of growth near $\partial\Omega$ coinciding with F in D^+.*

Proof. Let Problem 1 be solvable and u be its solution. The necessity of condition (10) is already proved. Set

$$\mathcal{F} = F - \chi_D u. \tag{11}$$

By the definition, $\mathcal{F}$ is harmonic in D^+. Then, by Martinelli-Bochner formula (7), Lemma 3 and Remark 1, we have:

$$\mathcal{F} = M_{\Omega} u_0 + T_{D,\Omega} f - \chi_D u = M_{\Omega}(\tilde{u}_0 - t(u)).$$

As $u_0 = t(u)$ on Γ then $M_{\Omega}(\tilde{u}_0 - t(u)) = M(\tilde{u}_0 - t(u))$ is harmonic in $\Omega \setminus \Gamma$ as parameter depending distribution. Hence $\mathcal{F}$ has the same property. It has a finite order of growth near ∂D because of the structure of the kernel $\mathfrak{U}(\zeta,z)$ and the compactness of ∂D: the kernel $\mathfrak{U}(\zeta,z)$ is harmonic outside the diagonal $\{\zeta = z\}$ and it grows as $|z-\zeta|^{1-2n}$ near the diagonal; besides, the compactness of ∂D implies that the distribution $(\tilde{u}_0 - t(u))$ has a finite order of singularity on ∂D.

Back, let there be a harmonic in Ω function $\mathcal{F}$ of finite order of growth near $\partial\Omega$ coinciding with F in D^+. Set

$$u = T_{D,D} f + M_D \tilde{u}_0 - \mathcal{F}^-. \tag{12}$$

As $f \in H(D, \Lambda^{(0,1)})$ and $u_0 \in \mathcal{D}'(\overline{\Gamma})$, Lemmata 2 and 3 imply that $T_{D,D} f + M_D \tilde{u}_0 \in H_{\overline{\partial}}(D)$. Moreover, since $D \subset \Omega$ then $\mathcal{F}$ is harmonic in D and has a finite order of growth near ∂D. Therefore $t(\mathcal{F}) \in \mathcal{D}'(\partial D)$ (see [4, Theorem 9.3.16]). Hence $\mathcal{F}^- = \mathcal{P}^{(D)}(t(\mathcal{F}))$ and $\mathcal{F}^- \in H_{\overline{\partial}}(D)$ because of Theorem 1. Thus, by the construction, the function u belongs to $H_{\overline{\partial}}(D)$. According to Corollary 1, there is boundary value $t(u)$ on $\overline{\Gamma}$ in the space of distributions which can be calculated by Definition 1.

Let take sequences $\{f_\nu\} \subset C^\infty(\overline{D}, \Lambda^{(0,1)})$ and $\{u_0^{(\nu)}\} \subset C^\infty(\partial D)$ approximating functions $f \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$ and $u_0 \in \mathcal{D}'(\partial D)$ respectively in these spaces. Then, because $T_D f_\nu \in C^2(\Omega)$ (see [4, Theorem 2.4.24]) and because of the Jump Theorem for Martinelli-Bochner integral (see, for instance, [14, Theorem 2.3 or Theorem 3.1]),

$$(T_D f_\nu)^- - (T_D f_\nu)^+ = 0 \text{ on } \Gamma, \quad (M\tilde{u}_0^{(\nu)})^- - (M\tilde{u}_0^{(\nu)})^+ = u_0^{(\nu)} \text{ on } \Gamma.$$

Now, using Stokes' formula and Lemmata 1, 3, 2 and Remark 1, we conclude that for all $g \in C^\infty_{comp}(D \cup \Gamma, \Lambda^{(0,1)})$ we have:

$$(\overline{\partial} u, g)_D - (u, \overline{\partial}^* g)_D =$$
$$= \lim_{\nu\to\infty} (\overline{\partial}(T_D f_\nu + M^- \tilde{u}_0^{(\nu)} - \mathcal{F}), g)_D - (T_D f_\nu + (M\tilde{u}_0^{(\nu)})^- - \mathcal{F}, \overline{\partial}^* g)_D =$$
$$= \lim_{k\to\infty} (u_0 + (T_D f_k)^+ + (M\tilde{u}_0^{(k)})^+ - \mathcal{F}), n(g)) = (u_0, n(g)),$$

i.e., $u = u_0$ on Γ.

In order to finish the proof we need to convince ourselves that $\overline{\partial} u = f$ in D. To this end consider the form $P = \chi_D(f - \overline{\partial} u)$ belonging to $H(\overline{\Omega}, \Lambda^{(0,1)})$. It is clear that $C^\infty_{comp}(\Omega, \Lambda^{(0,2)}) \subset C^\infty_{comp}(D \cup \Gamma, \Lambda^{(0,2)})$. Then, by (10) and Definition 1 we have for all $\beta \in C^\infty_{comp}(\Omega, \Lambda^{(0,2)})$:

$$(P, \overline{\partial}^* \beta)_\Omega = (f, \overline{\partial}^* \beta)_D - (\overline{\partial} u, \overline{\partial}^* \beta)_D = (u_0, n(*\overline{\partial}^* \beta)) - (u_0, n(*\overline{\partial}^* \beta)) - (u, \overline{\partial}^* \overline{\partial}^* \beta)_D = 0,$$

because $\overline{\partial}^2 \equiv 0$. Hence $\overline{\partial} P = 0$ in Ω.

On the other side, by Definition 1, for all $v \in C^\infty_{comp}(\Omega)$ we have:

$$(P, \overline{\partial} v)_\Omega = (f, \overline{\partial} v)_D - (\overline{\partial} u, \overline{\partial} v)_D = (f, \overline{\partial} v)_D - (u, \frac{1}{2}\Delta v)_D - (u_0, \overline{\partial}_n v). \tag{13}$$

Since $\mathcal{F}$ is harmonic in Ω and coincides with F in D^+, then (see Remark 1)

$$(\mathcal{F}, \frac{1}{2}\Delta v)_D = -(\mathcal{F}, \frac{1}{2}\Delta v)_{D^+} = -(T_{D,\Omega} f + M_\Omega \tilde{u}_0, \frac{1}{2}\Delta v)_{D^+}. \tag{14}$$

Besides, as Φ is bilateral fundamental solution to the Laplace operator in $\mathbb{C}^n$ then $\frac{1}{2}\Delta T_D f = \overline{\partial}^* \chi_D f$. Hence again taking a sequence $\{f_\nu\} \subset C^\infty(\overline{D}, \Lambda^{(0,1)})$, approximating $f \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$ in this space, and using Lemmata 1, 2 and Remark 1, we see that

$$(T_{D,\Omega} f, \frac{1}{2}\Delta v)_{D^+} + (T_{D,D} f, \frac{1}{2}\Delta v)_D = \lim_{\nu\to\infty} \left((T_D f_\nu, \frac{1}{2}\Delta v)_{D^+} + (T_D f_\nu, \frac{1}{2}\Delta v)_D \right) =$$

$$\lim_{\nu\to\infty}(T_D f_\nu, \frac{1}{2}\Delta v)_\Omega = \lim_{\nu\to\infty}(\overline{\partial}^* \chi_D f_\nu, v)_\Omega = \lim_{\nu\to\infty}(f_\nu, \overline{\partial} v)_D = (f, \overline{\partial} v)_D. \qquad (15)$$

Combining (13), (14), (15), we conclude that

$$(P, \overline{\partial} v)_\Omega = (M\tilde{u}_0, \frac{1}{2}\Delta v)_{D^+} + (M\tilde{u}_0, \frac{1}{2}\Delta v)_D - (u_0, \overline{\partial}_n v).$$

Finally, by the Stokes' formula, we have in the sense of weak limit values on Γ:

$$(M\tilde{u}_0, \frac{1}{2}\Delta v)_{D^+} + ((M\tilde{u}_0)^-, \quad \frac{1}{2}\Delta v)_D = ((M\tilde{u}_0)^- - (M\tilde{u}_0)^+, \overline{\partial}_n v)_{\partial D} +$$

$$+(\overline{\partial}_n((M\tilde{u}_0)^- - (M\tilde{u}_0)^+), v) = (u_0, \overline{\partial}_n v),$$

because in the sense of weak limit values on Γ there are the jumps on Γ:

$$(M\tilde{u}_0)^- - (M\tilde{u}_0)^+ = u_0, \quad \overline{\partial}_n((M\tilde{u}_0)^- - (M\tilde{u}_0)^+) = 0,$$

see [18] and [9] respectively.

Thus, $\overline{\partial}^* P = 0$ in Ω, and hence $(\overline{\partial} \oplus \overline{\partial}^*)P = 0$ in Ω. As every such a form has harmonic coefficients in Ω, the uniqueness theorem for harmonic functions yields $P \equiv 0$ in Ω. In particular, $f = \overline{\partial} u$ in D. □

Corollary 6. *Let $f \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$, $u_0 \in H^{-s-1/2}(\overline{\Gamma})$. The Cauchy Problem 1 is solvable in $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ if and only if (10) holds and there is harmonic in Ω function $\mathcal{F} \in H(\Omega, \|\cdot\|_{-s})$ coinciding with F in D^+.*

Proof. Indeed, if the Cauchy Problem 1 is solvable in $H_{\overline{\partial}}(D, \|\cdot\|_{-s})$ then it is solvable and $\mathcal{F} = M_\Omega(\tilde{u}_0 - u_0)$ (see the proof of Theorem 5). Hence, according to Lemma 3, function $\mathcal{F}$ belongs to $H(\Omega, \|\cdot\|_{-s})$.

Back, if $\mathcal{F} \in H(\Omega, \|\cdot\|_{-s})$ is harmonic and coincides with F in D^+ then the Cauchy Problem 1 is solvable. Therefore, its unique solution u is given by (12) and $\mathcal{F}$ is given by (11). In particular, $\chi_D u = (F - \mathcal{F}) \in H(\Omega, \|\cdot\|_{-s})$. Take $v \in C^\infty(\overline{D})$. Then there is $V \in C^\infty(\overline{\Omega})$ with $\|V\|_{s,\Omega} = \|v\|_{s,D}$ and $v = V$ in D. By the definition,

$$|(u, v)_D| = |(\chi_D u, V)_\Omega| \leqslant \|\chi_D u\|_{-s,\Omega} \|v\|_{s,D},$$

i.e., $u \in H(D, \|\cdot\|_{-s})$. Finally, as $\overline{\partial} u = f \in H(D, \Lambda^{(0,1)}, \|\cdot\|_{-s-1})$ we see that $u \in H_{\overline{\partial}}(D, \|\cdot\|_{-s})$. □

At the end, we note that Corollary 6 allows us to use the bases with double orthogonality property in order to construct Carleman's formulae for Cauchy Problem 1 in the same way as in [10] or [11].

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