FOUR GENERATED 4-INSTANTONS

I will present a joint work with Cristian Anghel, Iustin Coandă (arXiv:1604.01970). We show that there exist mathematical 4-instanton bundles F on the projective 3-space such that $F(2)$ is globally generated (by four global sections). This is equivalent to the existence of elliptic space curves of degree 8 defined by quartic equations. There is a (possibly incomplete) intersection theoretic argument for the existence of such curves in D'Almeida [Bull. Soc. Math. France 128 (2000), 577–584] and another argument, using results of Mori [Nagoya Math. J. 96 (1984), 127–132], in Chiodera and Ellia [Rend. Istit. Univ. Trieste 44 (2012), 413–422]. Our argument is quite different. We prove directly the former fact, using the method of Hartshorne and Hirschowitz [Ann. Scient. Éc. Norm. Sup. (4) 15 (1982), 365–390] and the geometry of five lines in the projective 3-space.

A mathematical n-instanton bundle on $\mathbb{P}^3$ (n-instanton, for short) is a rank 2 vector bundle F on $\mathbb{P}^3$, with $c_1(F) = 0$, $c_2(F) = n$, such that $\mathrm{H}^i(F(-2)) = 0$, $i = 0, \ldots, 3$. Examples of n-instantons are the bundles that can be obtained as extensions:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-1) \longrightarrow F \longrightarrow \mathscr{I}_{L_1 \cup \ldots \cup L_{n+1}}(1) \longrightarrow 0 \tag{1}$$

where $L_1, \ldots, L_{n+1}$ are mutually disjoint lines in $\mathbb{P}^3$. For $n \leq 2$, all n-instantons can be obtained in this way. This is no longer true for $n \geq 3$.

We are concerned with the problem of the global generation of twists of instantons. It is well known that if F is an n-instanton then F is n-regular hence $F(n)$ is globally generated. Gruson and Skiti showed that if F is a 3-instanton having no jumping line of maximal order 3 then $F(2)$ is globally generated. Our aim here is to prove the following:

Proposition 1. *There exist* 4*-instantons* F *on* $\mathbb{P}^3$ *such that* $F(2)$ *is globally generated.*

One shows that if F is a 4-instanton with $F(2)$ globally generated then $\mathrm{H}^0(F(1)) = 0$ and $\mathrm{H}^1(F(2)) = 0$ (hence $\mathrm{h}^0(F(2)) = 4$). It follows that the 4-instantons F with $F(2)$ globally generated form a nonempty open subset of the moduli space of 4-instantons.

We prove this proposition in an elementary way using the method of Hartshorne and Hirschowitz. The key point of our proof is the following:

Lemma 2. *Let* $L_1, \ldots, L_5$ *be mutually disjoint lines in* $\mathbb{P}^3$ *such that their union admits no* 5*-secant. Then there exist epimorphisms*:

$$\Omega_{\mathbb{P}^3}(1) \longrightarrow \mathscr{I}_{L_1 \cup \ldots \cup L_5}(3) \longrightarrow 0\,.$$

Acknowledgements. The special form of the morphisms $\sigma : \Omega_{\mathbb{P}^3}(1) \to \mathscr{O}_{\mathbb{P}^3}(3)$ used in the proof of Lemma 2 was "guessed" after several experiments using the progam `Macaulay2` of Grayson and Stillman.

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