# Bolzano's Concept of Consequence[1]

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(*The Monist* 85, 2002, 581-601)

## I Preliminaries

In the second volume of his *Wissenschaftslehre*[2] from 1837, the Bohemian philosopher, theologian, and mathematician Bernard Bolzano (1781-1848) introduced his concept of consequence, named *derivability* (*Ableitbarkeit*), together with a variety of theorems and further considerations. Derivability is an implication relation between *sentences in themselves* (*Sätze an sich*), which are not meant to be linguistic symbols but the *contents* of declarative sentences as well as of certain mental episodes. When Schmidt utters the sentence 'Schnee ist weiß', and Jones judges that snow is white, the sentence in itself expressed by Schmidt is the same as the one to which Jones agrees in thought. This sentence in itself is an abstract entity: in some sense, it exists; but it is unreal insofar as it lacks a position in space and time, does not stand in causal relationships, and is independent of the existence of thinking beings and languages.[3]

Sentences in themselves are conceived by Bolzano as the primary bearers of the unrelativized truth-values *true* or *false*. This should be understood as meaning, first, other things, such as sentences or judgements, have their truth-values in virtue of the truth-values of the sentences in themselves which are their contents. Second, what is expressed by, e.g., 'I am hungry' in each case does not possess relativized truth-values like *true/false with respect to person S at time t*. It includes elements specifying a particular time and person, which makes it unqualifiedly true or false. Third, there are neither truth-value gaps nor a third truth-value (e.g., *indeterminate*): every sentence in itself is either true or false.[4]

All in all, sentences in themselves are identical with, or at least similar to, Frege's *thoughts*.[5] I will frequently use the shorter term 'propositions', and I will refer to them by putting sentences in square brackets. [3 is a prime number] is the proposition expressed by '3 is a prime number'.

A sentence in itself consists of sub-propositional parts which Bolzano calls *ideas in themselves* (*Vorstellungen an sich*). [3 is a prime number] can be decomposed, among other things, into an idea of the number 3 and an idea of the property of being a prime number. These ideas in themselves are also neither

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[1] On the whole, this contribution is a summary of my book *Der Begriff der Ableitbarkeit bei Bolzano* (Siebel 1996). I would like to thank Wayne Davis and Wolfgang Künne for many valuable comments.

[2] I refer to it by 'WL' plus number of volume, section, and page. It is partly translated by Rolf George: *Theory of Science*, Oxford 1972; but here translations are mine.

[3] Cf. WL I, § 19, pp. 77f.; § 22, p. 90; § 25, p. 112; § 28, p. 121; WL II, § 122, 4.

[4] Cf. WL I, § 24, p. 108; § 25, p. 113; WL II, § 125, p. 7; § 147, pp. 77f.

[5] For a comprehensive comparison see Künne 1997.

lingustic symbols nor mental entities, but abstract objects which can become the contents of such things without existentially depending on them. In contrast to propositions, however, they are not true or false, but *empty* (*gegenstandlos*) if nothing falls under them, or *non-empty* (*gegenständlich*) if there is something which is represented by them.[6]

According to Bolzano, sentences in themselves are all structured in the same way. Whatever their linguistic counterparts look like, the expressed propositions have the form [*A* has *b*]. [*A*] is the *subject-idea*, which represents the object(s) the proposition is about; and [*b*] is the *predicate-idea*, which stands for the property (or properties) attributed to the object(s) (cf. WL II, § 127, p. 9). Hence, to display the structure and the constituents of a proposition as clearly as possible, one has to paraphrase the corresponding sentence into the canonical form '*A* has *b*'.[7] In many sections of the *Wissenschaftslehre*, Bolzano tries to do that for a number of sentences of different grammatical forms. For example, in sentences of the kind 'All *F* are *G*' or 'Every *F* is *G*' the quantifiers are regarded as not contributing an idea to the corresponding proposition but as just pointing out that the extension of '*F*' consists of *all F*s. The (somewhat awkward) paraphrase of 'All men are mortal' is 'Man has mortality' (cf. WL I, § 57, pp. 248f.). Apart from the end of section IV, however, I will ignore that issue.

Finally, Bolzano takes a proposition to be true only if its subject-idea is not empty (cf. WL II, § 127, p. 16). Since he does not allow for truth-value gaps, a proposition with an empty subject-idea is false. This holds even for sentences in themselves like [Round squares are round], and it results in Bolzano's acceptance of *conclusio ad subalternum*, i.e., the inference from 'All *F* are *G*' to 'Some *F* are *G*' (cf. WL II, § 155, p. 114).

## II The Definition of Derivability

Derivability is defined with the help of the *method of variation*, that is, the imaginary[8] substitution of ideas in a proposition (or an idea) by other ideas. By substituting them, we get variants of the original proposition (or idea) which may have a different truth-value (or extension). Replacing, e.g., [3] by [6] in [3 is a prime number] leads to a false sentence in itself, whereas additionally substituting [even number] for [prime number] results in the true variant [6 is an even number].

Implicitly or explicitly, Bolzano puts some constraints on variation. First, it has to be systematical, which means that same ideas must be replaced by same ideas. [Every Finnish man is drunk] is not a permissible variant of [Every tall man is tall]. Second, the variants must be non-empty, i.e., their subject-ideas have to represent at least one object (cf. WL II, § 147, p. 80). Substituting [the greatest prime number] for [3] in [3 is a prime number] is not allowed. It will soon become clear why Bolzano needs these requirements.

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[6] Cf. WL I, § 48, pp. 216-8; § 49, p. 220; § 54, pp. 237f.; § 66, p. 297.

[7] Cf. Textor 1997 for a criticism of that hypothesis.

[8] Bolzano himself has noticed that talk about substitution, as well as, e.g., products of a substitution, is metaphorical because sentences and ideas in themselves, as abstract entities, cannot be literally changed or generated (cf. WL I, § 69, p. 314).

The method of variation plays a key role in Bolzano's logic because he uses it to define a multitude of notions, such as logical truth, necessary truth, analyticity, and compatibility (*Verträglichkeit*). Since compatibility is a precondition of derivability, I briefly introduce its definition:[9]

> The propositions $P$ and $Q$ are compatible with respect to the variable ideas $I \leftrightarrow$ There is a substitution of $I$ which leads to true variants of $P$ and $Q$.

Thus, compatibility is not a two- but a three-place relation involving also the ideas which are taken as variable. For example, [6 is a prime number] is compatible with [8 is a prime number] with respect to [6] and [8] because replacing them by [3] and [5] generates true variants.

But note that, apart from the restrictions on the method of variation presented above, there is also a certain *liberality* which can easily be overlooked. The collection of propositions examined by variation may include propositions which do not contain the variable ideas in question.[10] In such borderline cases, the proposition itself is its one and only variant. Therefore, it is also allowed to take [6] as the only idea to be varied in order to ask whether [6 is a prime number] and [8 is a prime number] are compatible with respect to it. Regarding *this* idea, however, they are *not* compatible because [8 is a prime number] cannot be turned into a truth by substituting it. More generally, two propositions can be compatible with respect to certain ideas, whereas they are incompatible with respect to other ideas (cf. WL II, § 154, p. 101).

The same holds for derivability, whose definition reads as follows:

> [T]he propositions $M$, $N$, $O$, … [are] derivable from the propositions $A$, $B$, $C$, $D$, … with respect to the variable parts $i$, $j$, … if every collection of ideas which, in place of $i$, $j$, …, makes all the $A$, $B$, $C$, $D$, … true also makes all the $M$, $N$, $O$, … true. (WL II, § 155, 114)

Put simplistically, it must not happen that the variation leads to true variants of the premises and a false variant of one or more conclusions. This is a simplification, however, because Bolzano takes derivability to be a special kind of compatibility: the conclusions are derivable from the premises with respect to the variable ideas only if they are also compatible with respect to them.[11] Thus, the complete definition of Bolzano's concept of consequence amounts to:

> The propositions $Q$ are derivable from the propositions $P$ with respect to the variable ideas $I \leftrightarrow$ (i) There is a substitution of $I$ which leads to true variants of $P$ and $Q$, and (ii) every substitution of $I$ which leads to true variants of $P$ also leads to true variants of $Q$.

Hence, [Skippy is an animal] is derivable from [Skippy is a kangaroo] with respect to [Skippy]. The compatibility requirement is met, and replacing [Skippy] by other ideas never results in a true variant of the premise and a false variant of

[9] Cf. WL II, § 154, p. 100. '$P$' and '$Q$' stand for one or several propositions, '$I$' for one or several ideas.

[10] This is evident from a number of theorems of compatibility and derivability (cf. WL II, § 154, pp. 107f.; § 155, pp. 114f.; Bar-Hillel 1950-52, pp. 68f.; Morscher 1981, p. 116). In Siebel 1997, I point to some challenges of this liberality to Bolzano's definition of necessary truth.

[11] See the title of § 155. Furthermore, the compatibility clause is mentioned explicitly in WL II, § 248, pp. 474f., and it follows from the theorems in subsections 4, 16, 19, 20, 22, and 23 of § 155.

the conclusion. But if you also release the ideas [kangaroo] and [animal] for variation, the result is negative because condition (ii) is not satisfied. We have to bear in mind that derivability is also a triadic relation. A proposition may be derivable from another proposition regarding certain ideas without being derivable from it with respect to other ideas.

Furthermore, if we add [Skippy is not a kangaroo] as a second premise, and again consider [Skippy] as the only variable idea, the condition of compatibility is not satisfied anymore. For [Skippy is a kangaroo] and [Skippy is not a kangaroo] cannot be both made true by substituting [Skippy].

It should have become clear by now why Bolzano has to constrain the method of variation in the way mentioned in the beginning of this section. If we were allowed to substitute non-systematically, [Skippy is an animal] would not be derivable from [Skippy is a kangaroo] with respect to [Skippy] because replacing it in the premise by [Skippy's mother] and in the conclusion by [Canberra] would amount to a true variant of the former and a false variant of the latter. And if we took into consideration variants with an empty subject-idea, a triviality such as [Female kangaroos are female] would have many false variants, such as [Round squares are round]. But this contradicts Bolzano's claim that it belongs to the class of logically analytic propositions (cf. WL II, § 148, p. 84).

## III Characteristics of Derivability

There are some characteristics in virtue of which derivability differs from many modern conceptions of consequence. Three of them you know already: it is a *three-place* relation between *semantic contents* which requires *consistency*. Especially the first one – the incorporation of variable elements – is somewhat startling. After all, when we put forward arguments, we do not seem to mention anything to be varied. George (1983b, pp. 321-4), however, points out that Bolzano's account is not as strange as it might appear at first glance. In his opinion, the specification of variable parts provides the *form* of an argument; and we need to know its form in order to assess whether it is valid or not. Here is an example:

> Tom, Dick, and Harry are partners.
> Therefore, Tom and Dick are partners.

Is it a valid argument? That depends. A natural suggestion is that it is an argument which rests on the true assumption that, if three persons are *partners*, then also two of them are. This means to understand it as having the form:

> Persons *a*, *b*, and *c* are partners.
> Therefore, persons *a* and *b* are partners.

But it is also conceivable that someone presents it as an argument which is based on the false assumption that, if three persons constitute *a whole of some kind or other*, then also two of them do. That amounts to taking 'partners' as a further variable element, thereby assigning the argument the form:

> Persons *a*, *b*, and *c* are a whole of type *W*.
> Therefore, persons *a* and *b* are a whole of type *W*.

In the first case, the argument is valid, whereas, in the second case, it is not because three persons, but not two of them, can constitute a musical trio.

Hence, there is something to be said for Bolzano's account. The form of an argument need not always be specified explicitly by telling what is variable. We often realize from the context how we are to interpret it. But without knowing its form, it seems, we can hardly assess its validity.

The second characteristic – derivability's being defined for propositions – is due to the fact that Bolzano, like Frege, regards logic to be an "objective" science which is not concerned with human psychology or languages but with the realm of sentences and ideas in themselves (cf. WL I, § 16, pp. 61-6). Nevertheless, it is easy to apply Bolzano's definition to the level of linguistic signs. There are at least two ways to do that. The first was suggested by Bar-Hillel (1950-52, pp. 84f.) and Smart (1963, p. 563), and it consists in simply replacing 'sentences in themselves' and 'ideas in themselves' by 'sentences' and 'terms':

> The sentences $S_2$ are derivable from the sentences $S_1$ with respect to the variable terms $T \leftrightarrow$ (i) There is a substitution of $T$ which leads to true variants of $S_1$ and $S_2$, and (ii) every substitution of $T$ which leads to true variants of $S_1$ also leads to true variants of $S_2$.

This definition might attract one's attention because it comes close to a proposal which Tarski examines in his famous article "Über den Begriff der logischen Folgerung". (More about that in section V.) But it is far away from the spirit of Bolzano's tenets because, as I said, he takes sentences and ideas in themselves as the primary objects of logic. A definition which is in conformity with Bolzano's basic alignment should analyse derivability for sentences by recourse to the derivability of the propositions expressed by them:[12]

> The sentences $S_2$ are derivable from the sentences $S_1$ with respect to the variable terms $T \leftrightarrow$ The propositions expressed by $S_2$ are derivable from the propositions expressed by $S_1$ with respect to the ideas expressed by $T$.

Thus, in spite of the different subject terms, "Every duckling is an animal" is derivable from "Every little duck is a bird" with respect to "duckling" and "little duck" because they express the same idea.

The third characteristic – the compatibility clause – entails, first, that Bolzano's logic is *non-monotonic*. $Q$ may be derivable from $P$ with respect to $I$ without being derivable from $P$ and a further premise $R$ with respect to the same ideas. Adding a premise can violate the compatibility requirement, as is shown by an example already mentioned: [Skippy is an animal] is derivable from [Skippy is a kangaroo], but not from it plus [Skippy is not a kangaroo], with respect to [Skippy].

Second, the compatibility clause makes Bolzano's logic *non-contrapository*. $Q$ can be derivable from $P$ with respect to $I$ without $\neg P$ being derivable from $\neg Q$ with respect to these ideas. For $P$'s and $Q$'s being compatible regarding $I$ does not guarantee that $\neg P$ and $\neg Q$ are also compatible regarding these ideas.[13] Con-

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[12] The following definition should be taken merely as a starting point. For if a sentence (or term) is ambiguous, '*the* proposition (or idea) expressed' suffers a uniqueness failure. Davis (in personal communication) suggests to add a relativization to interpretations by talking about a sentence (or term) *interpreted as expressing a certain proposition (or idea)*.

[13] Cf. Bolzano's argument against transposition, i.e., the inference from *If p, then q* to *If $\neg q$, then $\neg p$* in WL II, § 248, pp. 478f.

sider [Socrates is a man] as the premise and [All men are men] as the conclusion. The conclusion is derivable from the premise with respect to [Socrates] and [man]: they are compatible, and there is no substitution leading to a true variant of the premise and a false variant of the conclusion, simply because the conclusion cannot be made false by replacing [man]. But the negation of [Socrates is a man] is not derivable from the negation of [All men are men] with respect to the ideas in question. The latter cannot be made true, therefore it is not compatible with the former.

Third, the condition of compatibility does not allow for *reductio ad absurdum*. In such inferences, a contradiction is derived from the assumptions in order to prove that they are inconsistent. For Bolzano, this procedure cannot even get going because there is nothing which is derivable from incompatible propositions, be it a contradiction or something else. This is a bit odd because Bolzano himself (e.g., in WL II, § 155, p. 117) uses *reductio ad absurdum*. In the fourth volume of the *Wissenschaftslehre* (§ 530), however, he recommends a method which, he claims, can transfer such proofs into proofs without incompatible premises. For the sake of brevity, I cannot examine it here. But if it works, Bolzano need not worry about the apparent discrepancy between his compatibility requirement and his own use of such inferences.

Finally, I want to emphasize a special feature of derivability which arises from something else. Today's logical systems find their bearings by the intuitive criterion that an argument is valid if and only if it is (*metaphysically / in principle*) *impossible* that its premises are true and its conclusion false. Thus, the argument

> Socrates is a man.
> Therefore, Socrates is a living being.

is valid because it is inconceivable that a man is not a living being. On the other hand, the argument

> Socrates is a man.
> Therefore, Socrates is at most 150 years old.

is not valid because, even if in fact there are none, we can at least imagine men who are older than 150 years.

The definiens of Bolzano's definition, however, contains no modal concepts. It does not state that the conclusion *must* be true if the premises are true, but merely that there is *de facto* no substitution which leads to true variants of the premises and a false variant of the conclusion. Thereby it gives free reign to inferences going beyond metaphysical necessity. If it is the case that human beings become at most 150 years old, then [Socrates is at most 150 years old] is derivable from [Socrates is a man] with respect to [Socrates], although there is no metaphysically necessary connection between being a man and being at most 150 years old.[14]

This shows that for Bolzano valid inferences can be based on laws of nature. In the sense of Ryle (1949, sect. V.2) and Toulmin (1953, sect. 3.8), he seems to view such laws as "inference tickets" which entitle to pass over from a premise to a conclusion without having to be included as a further assumption. Bolzano,

[14] Cf. also Bolzano's (1841, p. 56) own example of such an inference in his summary *Beurtheilende Uebersicht*.

however, goes beyond Ryle and Toulmin in allowing that the facts which legitimize such a transition need not even be laws of nature. Let us assume I possessed only punkrock records. Then [*The Buzzcocks* are a punkrock band] would be derivable from [M. S. possesses a record of *The Buzzcocks*] with respect to [*The Buzzcocks*] because there were no substitution of it making the premise true and the conclusion false. But it would not be a law of nature which ensures this.

## IV Logical Derivability

Bolzano's insistence on derivability being a triadic relation displays that he wants to capture a more general concept of consequence than modern logicians. Today, we are mostly interested in *formally* valid arguments, that is, arguments where the premises imply the conclusion because of their *logical* form. A case in point is the notorious:

> Socrates is a man.
> All men are mortal.
> Therefore, Socrates is mortal.

To recognize that this argument is valid, it suffices to know the meaning of its *logical* expressions. Bolzano, by contrast, wants to include *materially* valid arguments as well, i.e., arguments in which the meaning of *non-logical* expressions plays an important role. Under his general concept we can also subsume:

> Socrates is a man.
> Therefore, Socrates is mortal.

Such arguments are valid, but only materially valid. To assess their validity, one must know the meaning of the non-logical terms 'man' and 'mortal'. In Bolzano's logic this is reflected by the fact that there is derivability only if we take the non-logical ideas [man] and [mortal] as constant and thus [Socrates] as the only element which can be replaced and is therefore inessential.

But there are also some remarks in the *Wissenschaftslehre* making it clear that Bolzano is aware of the difference between material and formal validity. Thus, he writes:

> [There are] propositions which are *derivable* from [a proposition] *just in virtue of its form* (i.e., which are derivable if we conceive all parts in it as variable which logicians do not count among its *form*) [...]. (WL I, § 29, p. 141)

In a similar way, Bolzano demarcates a class of propositions, including, e.g., those of the type [*A* is *A*], which he calls *logically* analytic:

> [T]o assess the analytic nature of the former, we only need logical knowledge because the concepts which are constant in them all belong to logic [...]. (WL II, § 148, p. 84)

From these passages, we can distil a narrower conception of derivability which I name *logical* derivability. Its specific feature is that only the logical ideas are held constant, such that *all* non-logical ideas are free for substitution:

> The propositions $Q$ are logically derivable from the propositions $P \leftrightarrow Q$ are derivable from $P$ with respect to their non-logical ideas.[15]

For example, [Socrates is mortal] is logically derivable from [Socrates is a man] and [All men are mortal] because replacing their non-logical ideas [Socrates], [man] and [mortal] leads to a true variant of the conclusion whenever the variants of the premises are true.

The definition of logical derivability is based on the distinction between logical and non-logical items. Like Tarski (1936, p. 10) for a long time, Bolzano thought that this distinction cannot be fixed to the end of time (cf. WL II, § 148, p. 84). Presumably, this is a further part of his reason for focussing on three-place general derivability. If the complete description of an inference has to mention the variable parts, then the difference between logical and non-logical elements is irrelevant. It does not matter, then, whether they are cleanly separated or not.

Nonetheless, Bolzano offers some paradigms of logical ideas. It is tempting to count, as Berg (1981, p. 416) does, the meanings of quantifiers ('all', 'some', 'there is', …) and connectives ('and', 'or', …) among them. But this means modernizing Bolzano's account too much. In section I, I pointed out that he takes all propositions to have the same structure [$A$ has $b$]. Sentences of other grammatical forms, he claims, can be reformulated to reveal that structure.

Regarding the topic of logical ideas, it is interesting to see what happens with a sentence such as 'There is no square which is not a square' because it expresses a logical truth: the corresponding proposition is true solely in virtue of its logical elements, the other ideas can be replaced just as you like. According to Bolzano's remarks in §§ 137 and 138, it amounts to a statement about the emptiness of the idea [square which is not a square]. What the sentence states is more adequately described by 'The idea in itself of a square which is not a square has emptiness'. But then the only non-logical idea in the proposition is [square] because only that idea can be varied without getting a false proposition. Therefore, the other ideas must be logical, which includes not only [not] and [has] but also [idea in itself] and [emptiness].

We come to similar results when we look at Bolzano's paraphrases for conjunctions and disjunctions in § 160. Bolzano's logical ideas must not be identified with the meanings of those expressions we take as logical nowadays, especially quantifiers and connectives. Some of them (e.g., 'there is') do not even occur in the paraphrases, while such *meta*-ideas as [idea in itself] and [sentence in itself] are labelled logical – perhaps, because they represent logical objects (cf. WL II, § 223, p. 392). Analogously, Bolzano might view ideas like [emptiness] (and, presumably, [truth]) as logical because they represent properties of ideas or propositions.

## V Bolzano and Tarski

In section III, it was mentioned in passing that the definition of derivability for linguistic signs proposed by Bar-Hillel und Smart resembles a definition of logi-

[15] But see Morscher 1999 for a serious difficulty with that formulation and a suggestion for improvement. I cannot pursue that challenge here because that would lead to a paper too long.

cal consequence which Tarski examines in his article from 1936. It says that a sentence $S_1$ follows from sentences $S$ if and only if substitution of their extra-logical constants leads to a true variant of $S_1$ if the variants of $S$ are true (cf. Tarski 1936, p. 7). However, a comparison of these accounts would be more or less tedious because the former does not conform to the spirit of Bolzano's logic and the latter is rejected by Tarski. At best, the result were that Bolzano and Tarski agree on disapproving of the same definition.

It is illuminating, though, to have a look at Tarski's reason for rejecting the proposal. He realizes that the given condition is too weak for logical consequence because it can be satisfied just in virtue of there being not enough non-logical constants in the language in question in order to arrive at true variants of the premises and a false variant of the conclusion. It would be sufficient only if, e.g., there were singular terms for all objects; but, Tarski (1936, p. 7; my transl.) says, "this prerequisite is fictitious and cannot be realized". Therefore, as we will see soon, he passes over to varying objects instead of expressions.

But what about Bolzano's account? Does it circumvent that problem because it does not rest on variation of expressions but ideas? In Bolzano's view, the realm of ideas in themselves includes something analogous to singular terms for all objects: for every object, whether abstract or concrete, there is supposed to be a *singular idea*, that is, an idea representing nothing but it (cf. WL I, § 101, p. 470). Unfortunately, Cantor's diagonalization argument provides a serious challenge to that assumption, as Simons (1987, p. 402) made clear. Since Bolzano's ontology allows for such abstract objects as propositions and numbers, he would presumably also accept the sets of modern mathematics as further objects.[16] Hence, there should be a singular idea for every set. But diagonalization proves that the power set of a set, i.e., the set of all its subsets, contains more elements than the set itself. This means that the power set of the set of singular ideas had to include more elements than the set of singular ideas. Therefore, there would not be a singular idea for every object because there would not be a singular idea for every element in this power set. To reject the impact of this argument on his assumption, Bolzano had to show that there is no set of all singular ideas, or that it does not have a power set.

However, let us have a look at Bolzano's and Tarski's *official* definitions. Many philosophers and logicians claim a strong resemblance between them, and some even think that Bolzano anticipated Tarski's account.[17] In my view, this judgement is a bit exaggerated.

The central notion for Tarski's definition is *satisfaction of a sentential function*, where a sentential function is a chain of signs which results from a sentence by systematically replacing its non-logical constants by variables. Systematic substitution means here not only that same constants have to be replaced by same variables but also that different constants must be replaced by different variables (cf. Tarski 1936, p. 8). Furthermore, sentential functions are satisfied by sequences of objects (individuals or sets) which are assigned to the variables. For example, we can generate the sentential function '*Fa*' from the sentence 'Socra-

[16] With reference to Bolzano's notion of *collections* (*Inbegriffe*), some people regard sets as already being part of his ontology (cf., e.g., Berg 1992, p. 34). In my view, Bolzano's remarks about collections rather suggest that they are *mereological* wholes (cf. WL I, §§ 82-5).

[17] Cf. Hodges 1983, p. 56; Scholz 1953, p. 38; Smart 1963, p. 563; and also Tarski 1956, p. 417.

tes is a philosopher' by substituting the variable '*a*' for the singular term 'Socrates' and '*F*' for the general term 'is a philosopher'. Then we can assign, e.g., Mike Tyson to '*a*' and the set of boxers to '*F*'. This tupel satisfies the sentential function because Tyson is an element of the set of boxers. By contrast, the tupel consisting of Hegel and the set of boxers does not satisfy it because Hegel is not a member of that set.

Sequences of objects which satisfy a sentential function are called *models* of the corresponding sentence. Logical consequence is then defined as follows (cf. Tarski 1936, p. 9):

> The sentence $S_1$ follows logically from the sentences $S \leftrightarrow$ Every model of $S$ is also a model of $S_1$.

In other words, the conclusion follows logically from the premises if every sequence of objects satisfying the premise-functions also satisfies the conclusion-function.

How strongly is this connected with Bolzano's account? For a comparison it is natural not to take Bolzano's original definition of derivability but the definition of *logical* derivability as a starting point. There are two main differences between it and Tarski's definition. First, logical derivability is a relation between propositions, whereas logical consequence is defined for sentences. Second, in contrast to Bolzano's analysis, Tarski's does not require consistency. From sentences which have no model every sentence follows logically. But these differences can easily be removed. Let us transfer logical derivability to the realm of linguistic symbols in the way I have done it in section III for general derivability, and let us drop the condition of compatibility:

> The sentence $S_1$ is logically derivable from the sentences $S \leftrightarrow$ Every substitution of the non-logical ideas in the propositions expressed by $S$ and $S_1$ which leads to true variants of the propositions expressed by $S$ also leads to a true variant of the proposition expressed by $S_1$.

This explication comes close to Tarski's because both refer to a variation of extra-logical items: Bolzano varies the non-logical constituents in the expressed propositions, Tarski varies the objects which are assigned to the variables for the non-logical constants. Roughly, Bolzano's variation is a variation of *intensions*, whereas Tarski's is a variation of *extensions*. Furthermore, both relations are reflexive, transitive and neither symmetrical nor asymmetrical. Finally, they are both defined semantically because the concept of satisfaction of a sentential function is just as much semantical as the concept of expressing ideas and sentences in themselves.

However, despite this reconciliation, there remains an important difference: the Bolzanian relation is defined by recourse to semantic contents because those are Bolzano's primary logical objects. For that reason, there are sentences which are logically derivable from other sentences without following logically from them. Here is an example:

> Every duckling is a bird.
> Every bird is an animal.
> Therefore, every little duck is an animal.

If Tarski had said that we also have to replace *synonymous* non-logical constants by the same variable, we would be allowed to substitute the same variable for both 'duckling' and 'little duck'. But he prescribes that different constants must be replaced by different variables. Since 'duckling' and 'little duck' are different expressions, we thus get the following sentential functions:

Every $F$ is $G$.
Every $G$ is $H$.
Therefore, every $I$ is $H$.

Obviously, there are sequences of objects satisfying the premise-functions without satisfying the conclusion-function. Therefore, the conclusion does not follow logically from the premises.

Bolzano's definition, however, leads to another result. For Bolzano it does not matter that 'duckling' and 'little duck' are different terms. What is relevant are the *ideas* for which they stand. Since they express the same idea, say [duckling], the corresponding propositions are:

[Every duckling is a bird]
[Every bird is an animal]
[Every duckling is an animal]

By substituting the extra-logical ideas in these propositions, we never get true variants of the premises and a false variant of the conclusion. Hence, the conclusion is logically derivable from the premises.

In a nutshell, the common claim that Bolzano anticipated Tarski's definition of logical consequence must be qualified. First, it takes a number of steps to come from Bolzano's original explication, via logical derivability for propositions, to a definition which shares a greater number of features with Tarski's. Second, in spite of all similarities, there remains the difference that the Bolzanian relation has a larger extension because it does not differentiate between distinctly shaped but synonymous expressions.

## VI Bolzano and Relevance Logic

George (1983a, pp. 309-11) claims a further, less familiar, resemblance between Bolzano's and modern ideas. In his opinion, Bolzano partly anticipated *relevance logic*.

Classical propositional logic honours some arguments as valid which are at first glance strange. Among them is the inference from a contradiction '$p \,\&\, \neg p$' to an arbitrary conclusion '$q$' and the inference from an arbitrary premise '$p$' to a logical truth '$q \vee \neg q$'. According to classical logic, it is alright to conclude 'The moon is made of green cheese' from 'Socrates is a philosopher, and Socrates is not a philosopher' and to infer 'Tyson is a boxer, or Tyson is not a boxer' from 'Roses are red'. What is so strange about these so-called paradoxes of material (as well as strict) implication is that there is no connection in content whatsoever between premise and conclusion. Or, as relevance logicians say, the premises are not *relevant* for the conclusions.

But how to specify relevance in a formal way in order to get a logical system which does not allow for such inferences? The founders of relevance logic, Anderson and Belnap (1975, p. 33), propose as a necessary condition that the formulas share a variable:

> The formula $B$ is deducable from the formula $A \rightarrow$ At least one variable is contained both in $A$ and $B$.

Thus, the inference from '$p$ & $\neg p$' to '$p$' is unproblematic, whereas '$q$' is not deducable from '$p$ & $\neg p$' because they do not share a variable. In the same way, from a relevance logician's point of view, there is nothing to be said against the transition from '$p$' to '$p \vee \neg p$', but the inference from '$p$' to '$q \vee \neg q$' does not meet the condition of common variables.

There seems to be an analogue to that condition in Bolzano's *Wissenschaftslehre*:

> [If,] apart from the idea *has*, two propositions […] lack a common constituent, then it is obvious that, whatever ideas in these propositions we declare as variable, still there will never obtain a relation of derivability between them because the ideas […] are completely independent. (WL II, § 155, p. 120)

The other way round, this suggests:

> The proposition Q is derivable from the proposition $P$ with respect to the variable ideas $I \rightarrow$ At least one of the ideas $I$ is contained both in $P$ and Q.

This principle of common variable ideas indeed resembles very much the relevance logician's principle of shared variables. But a closer look at the Bolzanian counterparts to the paradoxes of material implication also reveals some differences.

Just as relevance logic agrees with the inference from '$p$' to '$p \vee \neg p$', so Bolzano has no objections to the claim, e.g., that [Socrates is a philosopher, or Socrates is not a philosopher] is logically derivable from [Socrates is a philosopher]. These propositions are compatible with respect to their non-logical ideas; substituting them does not lead to a true variant of the premise and a false variant of the conclusion because the latter will always be true; and every variable idea is contained both in the premise and the conclusion. Regarding the inference from '$p$' to '$q \vee \neg q$', the condition of shared variable ideas also entails the result we find in relevance logic. [Tyson is a boxer, or Tyson is not a boxer] does not appear to be logically derivable from [Socrates is a philosopher] because premise and conclusion do not share an extra-logical idea.

But things are different when we come to the transitions from '$p$ & $\neg p$' to '$p$' or '$q$'. In Bolzano's view, neither [Socrates is a philosopher] nor [Tyson is a boxer] is logically derivable from [Socrates is a philosopher, and Socrates is not a philosopher]. Although there are common constituents in the first case, it also fails because of the compatibility constraint. In both cases, the premise cannot be turned into a truth by replacing the extra-logical elements [Socrates] and [philosopher].

However, there is another point which is more important for a comparison between Bolzano's logic and relevance logic. Bolzano offers the condition of shared variable ideas as it were a *theorem* of his definition of derivability. But that is not true. Let us take again [Socrates is a philosopher] as our premise and [Tyson is a boxer, or Tyson is not a boxer] as the conclusion. According to the principle of shared variables, the latter is not logically derivable from the former. But if we consult Bolzano's official definition, there are no reasons against conceding derivability. The propositions are compatible with respect to their non-logical constituents; and there is no substitution of them bringing about a false variant of the conclusion, hence, *a fortiori*, every substitution resulting in a true variant of the premise leads to a true variant of the conclusion. Consequently, the latter should be logically derivable from the former.

All the more, Bolzano puts forward at least two claims about derivability violating the condition of common variable ideas. The first one is:

> If the propositions $A$, $B$, $C$, $D$, … are compatible with respect to the ideas $i$, $j$, …, whereas they are incompatible with the proposition $M$, then […] the proposition *Neg.* $M$ is derivable from them with respect to the same ideas. (WL, § 155, p. 118)

The propositions [Socrates is a philosopher] and [Aristotle is a philosopher] are compatible with respect to [Socrates] and [Aristotle], whereas they are incompatible with [6 is a prime number] regarding the same ideas. The latter does not contain these ideas, therefore it cannot be made true by replacing them (cf. WL II, § 154, p. 107). So, according to the principle above, the negation of [6 is a prime number] should be derivable from [Socrates is a philosopher] and [Aristotle is a philosopher] with respect to [Socrates] and [Aristotle]. But this does not conform to the constraint of common variables because the conclusion lacks the variable ideas and thus does not share one of them with the premises.

The second statement violating this constraint reads as follows:

> If the propositions $M$, $N$, $O$, … are derivable from $A$, $B$, $C$, $D$, … with respect to the greater number of ideas $i$, $j$, $k$, …, then they are also derivable with respect to the smaller number of ideas $j$, $k$, … (which are a part of the former), provided that the propositions $A$, $B$, $C$, $D$, … are compatible with respect to this smaller number of ideas […]. (WL, § 155, p. 119)

Consider [Every robin is a bird] and [Every bird is an animal] as the premises and [Every robin is an animal] as the conclusion. The conclusion is derivable from the premises with respect to [robin] and [bird], and the premises are compatible not only regarding these ideas but also with respect to the single idea [bird]. Hence, if the principle just mentioned were true, there would also obtain derivability with respect to that idea. But, again, this does not go well with the condition of shared variable ideas because the conclusion does not contain [bird].

This heavily calls into doubt that Bolzano's logic is subject to a relevance requirement. If Bolzano wants to stick to the principles above and his original definition of derivability, he better give up this constraint. Moreover, this does not amount to a serious interference because he can cling to what he claims before he introduces that condition:

> It is not the case that every proposition $M$ […] can be put into a relation of derivability with every proposition $A$ […] simply by taking as

> one likes ideas […] in these propositions as variable. (WL, § 155, p. 120)

It seems that the constraint of shared variables is just an unsuccessful attempt at substantiating this claim. Instead of basing it on that constraint, Bolzano could have also proven it by offering a concrete example. Just consider [Socrates is a boxer] and [3 is an even number]. Whatever ideas you release for substitution, these propositions will not stand in the relation of derivability with respect to them.

Thus, it is questionable whether Bolzano is a forerunner of relevance logic. Bolzano addicts might be disappointed by that result, but I think it helps us to understand what he was really getting at. Concerning Bolzano's aura as "the great anticipator", I totally agree with Morscher:

> We get nothing from acknowledging Bolzano's achievements always merely wholesale and dubbing him as a forerunner of so many tenets without getting to the bottom. This just raises unfounded hopes, which a down-to-earth historical appreciation is to destroy. (Morscher 1974, p. 103; my transl.)

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