ON THE ROLE OF FRUSTRATION AND NOISE IN DYNAMICAL SYSTEMS

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- Concept and Impact of Frustration in Dynamical Systems
- Frustration in Oscillatory and Excitable systems
- Frustration in the Context of Social Science and Computer Science

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FRUSTRATION IN SPIN SYSTEMS

+ ferromagnetic coupling
- antiferromagnetic coupling

Whatever position this spin takes, one of the neighbor bonds is frustrated.
IMPACT OF FRUSTRATION ON “ENERGY” LANDSCAPES

Particle in an attractive potential

Funnel energy landscape over configuration space. Are such shapes relevant for proteins or cell states?
WHAT IS FRUSTRATION IN SPIN SYSTEMS IS CURVATURE IN GENERAL RELATIVITY.

THE GENERIC CONCEPT OF PARALLEL TRANSPORT

Figures from Wikipedia
Parallel transport is needed to compare vectors at different points in curved space, here to compare matter fields at different space points due to the curvature of internal space. The parallel transport is provided by the connection $\Gamma$ or the gauge field $A$, respectively. The derivative which is sensitive to the intrinsic changes of vectors or matter fields is the covariant derivative, it vanishes if the vector is only parallel transported.
## Concept and Impact of Frustration in Dynamical Systems

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Conjecture concerning the impact of frustration:

Tuning an appropriate degree of frustration:

• not too low so that the dynamics is flexible enough and therefore functionally stable
• not too high, so that the dynamics is stable against noise

One of the “design principles” in evolutionary processes?
**CRITERION FOR FRUSTRATION IN COUPLED OSCILLATORY AND EXCITABLE SYSTEMS:**

- Consider a loop with interaction bonds and couplings that can be antagonistic: either ferromagnetic or antiferromagnetic, attractive or repulsive, excitatory or inhibitory, repressive or supportive.

- Consider a path from A to B along the shortest connection and along a complementary path in the loop from B to A.

**For undirected couplings:**

The bond from A to B is satisfied if A acts upon B in the same way as B upon A (e.g. attractive), otherwise it is not. If it is not, the loop from A to A is then frustrated.

**For directed couplings:**

The bond from A to B is satisfied if A acts upon B in the opposite way as B upon A (e.g. A to B activating, B to A via C and D repressing), otherwise it is not satisfied and the loop from A to A is frustrated.
Different realizations of frustration

via the combination of antagonistic couplings or the topology

e.g. A in phase with B, B with C → C with A, but if C wants to be antiphase with A, the link CA or CB is unsatisfied and the triangle is frustrated.
Quantitative criterion for frustration in case of phase oscillators

\[ \dot{\phi}_i = \omega + \frac{1}{N} \sum_{j=1}^{N} w_{ij} \sin(\phi_j - \phi_i) \quad i \in 1, \ldots, N \]


\[ F = -\frac{1}{2N} \sum_{i,j=1}^{N} w_{ij} \cos(\phi_j - \phi_i) \]

Quantitative measure

\[ \dot{F} = \sum_{i=1}^{N} \frac{\partial F}{\partial \phi_i} \dot{\phi}_i = -\sum_{i=1}^{N} (\frac{\partial F}{\partial \phi_i})^2 \leq 0 \]

Qualitative measure:

\[ w(i \ j) = +1 \quad \text{or} \quad +1 \]

Kuramoto dynamics drives the system in local minima of the frustration landscape, depending on the initial conditions.
FRUSTRATION IN OSCILLATORY AND EXCITABLE SYSTEMS

Individual dynamics here

- Phase oscillators: Kuramoto oscillators and classical rotators
- Motifs from genetic circuits: Bistable frustrated units,

coupled on regular grids
FRUSTRATED SETS OF CLASSICAL ROTATORS AND KURAMOTO OSCILLATORS

Without noise:
- Extremely rich attractor space
- Spontaneous onset of global couplings
- The Watanabe-Strogatz phenomenon, dynamically generated
- Spontaneous generation of slow dynamics for small perturbations
- Physical aging for natural frequency distributions

Under the action of noise:
- Order-by-disorder phenomenon
- Hints on hierarchies in the potential landscape
- Noise-driven migration of oscillator phases through the attractor landscape
- Physical aging
OUTLINE: IN ANALOGY TO SPIN SYSTEMS WE SHALL SEE HOW

- frustration
- multistability
  - rough "energy" landscape with a hierarchy of barriers
  - multitude of inherent time scales
    - aging

here in excitable and oscillatory systems with and without noise
CLASSICAL ROTATORS WITH FRUSTRATION

The model: N active rotators with repressive coupling

\[ \frac{d \varphi_i}{dt} = \omega_i - b \sin \varphi_i + \sigma_A \xi_i(t) + \left( \kappa + \sigma_M \eta_i(t) \right) \sum_{j} A_{ij} \sin(\varphi_j - \varphi_i). \]

Fig. 1: Hexagonal lattice with all triangles frustrated for all couplings being negative (a) and not frustrated (b) for positive couplings along the horizontal links and negative ones otherwise.
The versatility of attractors in comparison to spin systems

In spin glasses:

- fixed points

In these systems:

- various combinations of fixed points
- a variety of limit cycles differing by their:
  - correlation between individual phases
  - frequency
  - pattern of phase-locked motion
  - basin of attraction
  - stability
  - symmetry
- quasiperiodic solutions
- chaotic solutions

as a combined effect of frustration, lattice size and lattice symmetry.
EXAMPLES FOR PHASE-LOCKED MOTION IN PARTICULAR PLANE WAVES AND SPHERICAL WAVES:

We expected to see order-by-disorder phenomena in analogy to spin systems.

Fig. 2: Solutions of eq. (1) with 4 and 6 clusters on a $4 \times 4$ lattice for $\omega = 0.7$, $b = 1$, $\kappa = -2$ and $\sigma_M = \sigma_A = 0$. 

We expected to see order-by-disorder phenomena in analogy to spin systems.
Phase Space without Noise:

**Self-organized Strogatz-Watanabe phenomenon** (M. Zaks, P. Tomov)

Network of $N$ *identical* one-dimensional elements.

$$\frac{d\varphi_i}{dt} = f(\varphi_i) + \kappa \sum_j A_{ij} g(\varphi_i, \varphi_j)$$

with adjacency matrix $A_{ij}$.

- Global coupling: synchronization, in case of coupling through the first Fourier harmonics: Strogatz-Watanabe phenomenon (existence of $N$-3 conserved quantities).

Here $N=4 \Rightarrow$ frequency continuum of 4-cluster states of 16 oscillators, globally coupled
In view of the application of noise: Short excursion from
The Impact of Stochastic Fluctuations in Physics and Biology

The role of noise in dynamical systems

Destructive:
- if bad signal-to-noise ratio
- loss of synchrony in circadian clocks
- lower precision of signals

Constructive:
- if it improves the signal-to-noise ratio such as in
  - stochastic resonance,
  - coherence resonance and
  - system size resonance

Noise in nonlinear systems and out-of-equilibrium can act in a way that appears as counterintuitive or is not foreseeable.
Coherence resonance:
For an intermediate strength of noise, oscillatory response in an excitable system is most coherent (here no external field)

Stochastic resonance:
Under the application of a deterministic periodic external field an optimal signal-to-noise ratio is achieved for an intermediate noise level and

System size resonance
For an optimal size of the system, the system’s response is most regular, illustrated in a system of coupled nonlinear noisy oscillators (ensemble averages fluctuate with \((D/N)^{1/2}\))

\[D\] the (effective) noise intensity
Results for Classical Oscillators with Noise:

The Order-by-Disorder Phenomenon

Usually: **Order-by-disorder is considered in spin systems.**

**Generic:** The ground state is degenerate due to competitions among the interactions. The degeneracy is lifted due to disorder.

The lifting can be

\[
\begin{cases}
\text{or quantum driven (Chubukov, PRL (1992), Reimers et al. PRB (1993))} \\
\text{or due to dilution (Henley PRL 1989)}
\end{cases}
\]

Here: the more oscillator phases coalesce, the higher the order, disorder introduced by additive or multiplicative noise
In view of the order-by-disorder phenomenon:

The classification according to $p_n$-patterns with $n$ denoting the number of clusters of coalescing phases is not unique, but suited for our notion of order.

- **p4-solutions**: 4 clusters, originally only 75 such solutions could be identified, differing by their frequency, in particular plane waves (without counting the degeneracy due to the lattice symmetries)

- **p6-solutions**: spherical waves (degeneracy 96 due to 16 sites for the center and 6 rotations about 60 degrees due to the lattice symmetry)

- **p16-solutions** of individual limit cycles

- **Quasiperiodic solutions**

using the numerically obtained Poincaré-mapping on the hypersurface $\Phi_1 = \text{const.}$
lattice of 10x10 active rotators
clusters 100—10--100

Varying the noise intensity between 0.0001-0.1
Zoom into the noise intensity:

Increasing monotonically the noise intensity from left to right and top to bottom one observes disorder (d) and order (o) as the sequence:

\[ d \ o \ d \ o \ d \ o \ d \ o \ d \]

The system is very sensitive to the initial condition.

Varying the noise intensity between 0.001-0.1 in steps of 0.01

lattice of 10x10 active rotators
clusters 100—10—100—10—100—10---100
Zoom further into the noise intensity:

Increasing monotonically the noise intensity from left to right and top to bottom, one observes disorder (d) and order (o) as the sequence:

d o d o d

lattice of 10x10 active rotators
clusters 100—10—100—10—100

Varying the noise intensity between 0.06-0.07 in steps of 0.001
How representative are these plots? Similar plots are obtained

- if the noise realization is varied
- the additive noise is replaced by multiplicative one
- the oscillators are reduced to Kuramoto oscillators
- the lattice size is increased
- the time windows for snapshots over 200 time units are varied.
What is the explanation for the repeatedly increasing order when the noise strength is monotonically varied?

The system obeys a gradient dynamics with potential $V$. The resolution of its shape depends on the noise intensity

$V = -\omega \sum_i \varphi_i - b \sum_i \cos \varphi_i - \frac{\kappa}{2N} \sum_{i,j} A_{ij} \cos(\varphi_j - \varphi_i)$. 

- hierarchy in potential barriers
Noise-driven Migration of Oscillator Phases through a Complex Landscape:

Keeping the noise intensity fixed we see the following kind of "state"

\[ V_{\text{osc}} \]

\[ \Phi_i(t) \]

0-200 time units
The state is characterized by ongoing transitions between the different pattern of phase locked motion, characterized as p4, p6, p16, disordered, or transient p3 states.

9600-9800 and so on for ever

We see a multitude of escape times between the metastable states.
SUMMARY SO FAR

Active rotators and Kuramoto oscillators on a hexagonal lattice with frustrated bonds

- show a large number and a variety of coexisting attractors.
- can exhibit a spontaneous generation of global couplings and the W.-S.-phenomenon with a continuum of solutions.
- Under noise we see the ongoing migration of phases through the potential landscape.
- The escape times between the metastable states define a multitude of time scales.

We expect to see aging of these oscillators.
Defining criteria for physical aging (in contrast to biological aging):

Do relaxation processes towards the stationary state show

- slow dynamics
- breaking of time-translation invariance
- dynamical scaling?
MEASURE AGING OF ACTIVE ROTATORS AND KURAMOTO OSCILLATORS

via the autocorrelation functions:

- Prepare the system in the vicinity of the unique fixed point at $\kappa > 0$.
- Quench the system towards $\kappa < 0$ in the regime of coexisting synchronized oscillations to push it “out-of-equilibrium”.
- Wait and let it evolve under the action of additive noise.
- Perform a first measurement of the autocorrelation function at time $t_w$.
- Perform a second measurement at time $t > t_w$

for two lattice sizes (32x32 and 4x4) and three noise intensities $\sigma = 0.01, 0.1, 0.5$ with and without frustration.

Here the coupling provides the bifurcation parameter from one phase into the other and the noise creates the fluctuations.
The state of the system at time $t$ is specified by the vector of all phases

$$\vec{\phi} = (\Phi_1(t), \Phi_2(t), \ldots, \Phi_N(t)).$$

We compute the two-time autocorrelation function defined as

$$C(t, t_w) := \frac{\langle \vec{\phi}(t)\vec{\phi}(t_w) \rangle - \langle \vec{\phi}(t) \rangle \langle \vec{\phi}(t_w) \rangle}{\sigma_t \sigma_{t_w}},$$

with standard deviations $\sigma^2_t = \langle \vec{\phi}(t)\vec{\phi}(t) \rangle - \langle \vec{\phi}(t) \rangle \langle \vec{\phi}(t) \rangle$,

where the averages are calculated over a sufficient number of noise realizations.
AGING OF ACTIVE ROTATORS: 4X4 AND 32X32 WITH NOISE

Regime (i) Quasiequilibrium

Regime (ii) Drop off

Regime (iii) Slow saturation
Dynamical Scaling of Active Rotators for 4x4 and 32x32

\[ t_w^{b_1} C(t, t_w) = f\left(\frac{t}{t_w}\right) \]

\[ b \in [0, 0.02] \ (4 \times 4) \text{ and } b \in [0, 0.03] \ (32 \times 32) \]
So no dependence on the waiting time $t_w$, neither for the 4x4 case.
The mechanism seems to be the same as for spin glasses, but the attractor landscape is much more versatile, in which the phases continue to move from one metastable state to another, kicked by noise.
What are different aging mechanisms?

What is the role of noise in aging? (Montemurro et al. PRE67,031106(2003))

Replace noise by “disorder” in the natural frequencies for Kuramoto oscillators with repulsive coupling with deterministic or random realizations of the frequency distributions (ongoing work with D. Labavic)
Consider again the Kuramoto model on a hexagonal lattice,

\[
\frac{d\phi_i}{dt} = \omega_i + \frac{\kappa}{6} \sum_j A_{ij} \sin(\phi_j - \phi_i)
\]

but now with a distribution in natural frequencies according to

\[
g(\omega) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(\omega-\mu)^2}{2\sigma^2}}
\]

where we either

- I. choose $g_i$ randomly from $g(\omega)$ for all sites, or
- II. deterministically such that the maximal $\omega_i$ is in the center of the grid and the minimal ones at the boundaries, otherwise $\mu = 1$ and $\sigma = 0.01$ comparable to Kuramoto with $g(\omega) = \omega$. 

Aging in Deterministic Systems
Regular frequency distribution with the largest in the center
1. Choosing the protocol as before by starting from a state, where all phases are the same and then choosing the coupling deeply in the multistable regime, measuring the autocorrelation time as before, we observe

4x4-lattice with I. random selection of $\omega$ according to $g(\omega)$, averaged over different realizations of $g(\omega)$
The role of disorder is similar to the role of noise w. r. t. the phase evolution in time: the synchronization patterns change as if the system explores a rich landscape.

\[ \sin \Phi_i \]

4x4 lattice random distribution of \( \omega \) with \( \mu = 1 \) and \( \sigma = 0.01 \)

i) 0-200, ii) 10000-10200, iii) 50000-50200 t.u.
but in contrast to being randomly kicked from one metastable state to the next, here the different intermediate metastable states are visited periodically after very long periods, as it can be seen from the order parameter:

$$|Z|$$

2000 t.u. in contrast to a period of 10 t.u. in the period of oscillator phases

$$Z(t) := |Z|e^{i\theta} = \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j}$$
II. Deterministic realization of frequencies for a 4x4 lattice:

Aging in the autocorrelation function with averages over different initial conditions

4x4-lattice with II. deterministic selection of $\omega$ according to $g(\omega)$, averaged over different initial conditions only
Deterministic implementation of the natural frequency distribution:
Order parameter for 3 slightly different initial conditions after 50000 t.u.

Periods of the order parameter between 500 and 5000 t.u.

|z|

⇒ obvious multistability
Consider in more detail individual phase trajectories, and choose in parallel a representation on the unit circle.
What do the phase trajectories look like for selected points of the order parameter?
Heteroclinic orbits connecting unstable limit cycles?
• Notice: we have an emergence of very long time scales out of short time scales from the individual oscillations.

• The rather rich attractor space is now deterministically explored along very long closed paths, depending on the initial conditions and/or on the realization of the random distribution of $g(\omega)$.
Figure 4. Histograms of order parameter periods as a function of the period length for 500 realizations of $g(\omega)$ for random $\omega$’s, and 500 different initial conditions for regular $\omega$’s for two different lattice sizes. Parameters are $\mu = 1$, $\sigma = 0.01$, $\kappa = -2$, initial conditions are taken from a vicinity of the fixed point at $\varphi_i = 0.5$. 
Both the period of the heteroclinic cycle and the transient time towards it increase with decreasing width of regular frequency distributions.

What is the influence of the width of $g(\omega)$?
Self-similar temporal sequences

\[ \sigma = 0.001 \]
\[ \sigma = 0.002 \]
\[ \sigma = 0.003 \]
\[ \sigma = 0.004 \]
\[ \sigma = 0.005 \]
\[ \sigma = 0.006 \]

Notice: \[ \sigma t = 48.4 \pm 0.9 \]
In particular we have long transients.

Figure 2. Transients of the order parameter for initial conditions around the fixed point. Two trajectories in black and red are shown for a $4 \times 4$ hexagonal lattice for random $\omega$’s, corresponding to two different realizations of $g(\omega)$ with the same mean $\mu$ and standard deviation $\sigma$, and regular $\omega$’s, corresponding to two different initial conditions, respectively. All initial conditions are taken from a small radius around $\varphi_i = 0.5 \ \forall i$. The parameters are $\mu = 1$, $\sigma = 0.01$, and $\kappa = -2$. 
AGING MECHANISM IN DETERMINISTIC SYSTEMS:

- Rich attractor space with long and versatile inherent time scales that got explored formerly via noise and now during the deterministic time evolution.

- Depending on the waiting time, a different part gets explored, with which the second measurement in the autocorrelation function correlates.

- The smaller the width of the distributions, the longer on average the periods of the heteroclinic orbit.

OUTLOOK TO OPEN QUESTIONS AND NEXT STEPS

- Other manifestations of aging in the response to external forces
- Physical aging in other oscillatory systems like genetic circuits

Any relation to biological aging?

*physical aging* in the sense of an age dependent response to perturbations
*biological aging* e.g. in the sense of a deterioration of pacemaker cells.
So far to oscillatory and excitable systems, but also to:

- **social science** and the approach of social balance
- **computer science** and satisfiability problems
  

REDUCTION OF FRUSTRATION IN “SPIN-LIKE” SOCIAL SYSTEMS

Social balance and triad dynamics on various given topologies

Network: all-to-all topology
Nodes: individuals
Links: +1 for friends, -1 for enemies

Antal et al., PRE 72,036121 (2005)

Triad dynamics to approach a balanced state (the “paradise”)

\[
\begin{align*}
& \text{i} \quad \text{k} \\
\rightarrow & \quad \text{p} \\
\leftarrow & \quad (1-p) \\
\end{align*}
\]

imbalanced \quad frustrated \quad balanced

imbalanced \quad balanced

p is the propensity parameter driving a certain phase transition
Consider first artificial networks with triad dynamics where the frustration can be reduced even to zero.

**Questions of interest here:**

- What is the time to reach the frustration-free or balanced states?
- Can a local stochastic algorithm find these states in a finite time?
- How does the time depend on the algorithm that is used?


- **From triad to k-cycle dynamics** (distinguish between even and odd k)
- From all-to-all to random, diluted and to regular topologies
- **Mapping to optimization problems in computer science**
Comparison between the approach to social balance and SAT

**Different:** meaning of frustration

**In common:** formal criterion for frustration
algorithm driven by the reduction of frustration via
flipping signs
solution of the problem exists (here ferromagnetic
state). Question: how to find it?

Is it possible via a local stochastic algorithm within a finite time?

Whether yes or no depends on

- the propensity parameter $p$
- the degree of dilution $\alpha$
- the network topology
SUMMARY AND CONCLUSIONS

Concept of frustration is a very generic one. It

• can explain the proliferation of attractors and effects going along with frustration like order-by-disorder or aging in oscillatory or excitable systems

• has implications for control and design to keep a device flexible to avoid conflict like geometric frustration in power grids

• leads to interesting perspectives for out-of-equilibrium physics in a broader class of systems with possible applications to biology and beyond
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