Negative-weight percolation

A.K. Hartmann\textsuperscript{1}, O. Melchert\textsuperscript{1}, and L. Apolo\textsuperscript{2}

\textsuperscript{1} Institut für Physik, Universität Oldenburg
\textsuperscript{2} City College of the City University of New York

APS March Meeting, 15 March 2010
Outline

- Percolation problem
- Algorithms
- Results: two-dimensional, higher dimensions
- Summary
Agent travels:
“$I$ want to get from $A \rightarrow B$”

$\leftrightarrow$ standard (connectivity) percolation
Motivation

Agent travels:
“I want to get from A → B”
↔ standard (connectivity) percolation

Agent travels:
pays for travel resources (positive)
can earn resources (negative payment)
“I want to make a profit going from A → B”
↔ negative-weight percolation
Model

- $L \times L$ lattice, fully periodic boundary conditions
- Undirected edges, weight (cost) distribution:

$$P(\omega) = \rho \left(2\pi\right)^{-1/2} \exp\left(-\omega^2/2\right) + (1-\rho) \delta(\omega - 1)$$

- Allows for loops $\mathcal{L}$ with negative weight $\omega_\mathcal{L}$
- Agent on lattice edges: pay/receive resources

- Configuration $\mathcal{C}$ of loops, with

$$E \equiv \sum_{\mathcal{L} \in \mathcal{C}} \omega_\mathcal{L} \overset{!}{=} \min_{\mathcal{L} \in \mathcal{C}}$$

- Obtain $\mathcal{C}$ through mapping to minimum weight perfect matching problem [O. Melchert & AKH, New J. Phys. 2008]
\[ d(i) = \min_{j \in N(i)} (d(j) + \omega(i, j)) \] is not fulfilled
\[ d(i) = \min_{j \in N(i)} (d(j) + \omega(i, j)) \] not fulfilled
\[ d(i) = \min_{j \in N(i)} (d(j) + \omega(i, j)) \] not fulfilled
Minimal distances

\[ d(i) = \min_{j \in N(i)} (d(j) + \omega(i, j)) \] not fulfilled

Standard minimum-weight path algorithms, e.g. Dijkstra, Bellman-Ford, Floyd-Warshall, don’t work
\[d(i) = \min_{j \in N(i)} (d(j) + \omega(i,j))\] not fulfilled

Standard minimum-weight path algorithms, e.g. Dijkstra, Bellman-Ford, Floyd-Warshall, don’t work

Minimum-weight path problem requires matching techniques

[R.K. Ahuja, T.L. Magnanti and J.B. Orlin, *Network flows*]
Algorithm – Outline

Brief description of the basic steps:

- Set up auxiliary graph
- Find minimum weight perfect matching (MWPM) on auxiliary graph
- Interpret MWPM as minimal weighted set of paths/loops

Graph $G = (V, E)$:
Algorithm – Outline

Brief description of the basic steps:
- Set up auxiliary graph
- Find minimum weight perfect matching (MWPM) on auxiliary graph
- Interpret MWPM as minimal weighted set of paths/loops

Matching $M \subset E$: 

![Diagram of a graph with weighted edges and vertices labeled from 0 to 7]
Brief description of the basic steps:

- Set up auxiliary graph
- Find minimum weight perfect matching (MWPM) on auxiliary graph
- Interpret MWPM as minimal weighted set of paths/loops

MWPM, $\omega_M = 2$
Algorithm – Mapping procedure

(a) (b)

(c) (d)

+1 +1 +1 +1
-1 -1 -1 +1
+1 -1 -1 -1
+1 +1 +1 +1
Observe system spanning loops above critical $\rho$

Disorder induced, geometric transition

Characterize loops using observables from percolation theory (finite-size scaling (FSS) analysis)

[D. Stauffer, A. Aharony, *Introduction to Percolation Theory*]
Percolation probability exhibits FSS:

\[ P_L^s \sim f[(\rho - \rho_c)L^{1/\nu}] \]

\( \rho_c = 0.340(1) \)
\( \nu = 1.49(7) \)

(rand. perc.: \( \nu = 1.33 \))
\( S = 0.91 \)

- \( S = \) “quality” of the scaling assumption
- Similar scaling for mean number of spanning loops
- Compatible results for spanning negative-weight paths.
Probability \( P_L^\infty \equiv \langle \ell \rangle / L^d \) that edge belongs to percolating loop, finite-size susceptibility \( \chi \equiv L^{-d}(\langle \ell^2 \rangle - \langle \ell \rangle^2) \)

Exhibits FSS:

\[
P_L^\infty \sim L^{-\beta/\nu} f[(\rho - \rho_c) L^{1/\nu}]
\]

\[
\beta = 1.07(6)
\]

\[
S = 1.16
\]
Probability $P_L^\infty \equiv \langle \ell \rangle / L^d$ that edge belongs to percolating loop, finite-size susceptibility $\chi \equiv L^{-d}(\langle \ell^2 \rangle - \langle \ell \rangle^2)$

At $\rho_c$ ($L_{\text{max}} = 512$):
- loop length $\langle \ell \rangle \sim L^{d_f}$,
- roughness $\langle r \rangle \sim L^{d_r}$,
- suscept. $\chi \sim L^{\gamma/\nu}$

$d_f = 1.266(2)$
$d_r = 1.001(4)$
$\gamma = 0.77(7)$

Scaling relations $d_f = d - \beta/\nu$ and $\gamma + 2\beta = d\nu$ are fulfilled
Fisher exponent

Distribution $n_\ell$ of the loop lengths $\ell$ at $\rho_c$ for $L = 256$

Expected FSS:

$$n_\ell \sim \ell^{-\tau}$$

$$\tau = 2.59(3)$$

- Excluding spanning loops
- Consistent with scaling relation $\tau = 1 + d/d_f$
High dimensions

Average loop length

Loop length distribution

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho_c$</th>
<th>$\nu$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$d_f$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.340(1)</td>
<td>1.49(7)</td>
<td>1.07(6)</td>
<td>0.77(7)</td>
<td>1.266(2)</td>
<td>2.59(3)</td>
</tr>
<tr>
<td>3</td>
<td>0.1273(3)</td>
<td>1.00(2)</td>
<td>1.54(5)</td>
<td>-0.09(3)</td>
<td>1.459(3)</td>
<td>3.07(1)</td>
</tr>
<tr>
<td>4</td>
<td>0.0640(2)</td>
<td>0.80(3)</td>
<td>1.91(11)</td>
<td>-0.66(5)</td>
<td>1.60(1)</td>
<td>3.55(2)</td>
</tr>
<tr>
<td>5</td>
<td>0.0385(2)</td>
<td>0.66(2)</td>
<td>2.10(12)</td>
<td>-1.06(7)</td>
<td>1.75(3)</td>
<td>3.86(3)</td>
</tr>
<tr>
<td>6</td>
<td>0.0265(2)</td>
<td>0.50(1)</td>
<td>1.92(6)</td>
<td>-0.99(3)</td>
<td>2.00(1)</td>
<td>4.00(2)</td>
</tr>
<tr>
<td>7</td>
<td>0.0198(1)</td>
<td>0.41(1)</td>
<td>–</td>
<td>–</td>
<td>2.08(8)</td>
<td>4.50(1)</td>
</tr>
</tbody>
</table>

$\rightarrow$ upper critical dimension = 6

Summary

- Negative-weight percolation of loops
- Distinct from random bond/site percolation
- $2d$: critical exponents close to RBIM
- Upper critical dimension: 6
- More details:
Negative-weight percolation of loops
Distinct from random bond/site percolation
$2d$: critical exponents close to RBIM
Upper critical dimension: 6
More details:

Thank you for your attention!

New book (do better simulations):