The Fechnerian Idea

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From the principle that subjective dissimilarity between 2 stimuli is determined by their ratio, Fechner derives his logarithmic law in 2 ways. In one derivation, ignored and forgotten in modern accounts of Fechner's theory, he formulates the principle in question as a functional equation and reduces it to one with a known solution. In the other derivation, well known and often criticized, he solves the same functional equation by differentiation. Both derivations are mathematically valid (the much-derided "expedient principle" mentioned by Fechner can be viewed as merely an inept way of pointing at a certain property of the differentiation he uses). Neither derivation uses the notion of just-noticeable differences. But if Weber's law is accepted in addition to the principle in question, then the dissimilarity between 2 stimuli is approximately proportional to the number of just-noticeable differences that fit between these stimuli: The smaller Weber's fraction the better the approximation, and Weber's fraction can always be made arbitrarily small by an appropriate convention. We argue, however, that neither the 2 derivations of Fechner's law nor the relation of this law to thresholds constitutes the essence of Fechner's approach. We see this essence in the idea of additive cumulation of sensitivity values. Fechner's work contains a surprisingly modern definition of sensitivity at a given stimulus: the rate of growth of the probability-of-greater function, with this stimulus serving as a standard. The idea of additive cumulation of sensitivity values lends itself to sweeping generalizations of Fechnerian scaling.

Gustav Theodor Fechner's principal work, *Elemente der Psychophysik,* turned 150 years old in 2010. From the publication of this book many date the beginnings of scientific psychology. By the mid-19th century the Enlightenment tradition had long since made the adjective *scientific* synonymous with *physics-like*. That is, the *scientific* implied systematic measurements informing a mathematical theory and being guided by it, the theory itself consisting of postulated laws and their logical consequences. Therefore, the term *psychophysics,* coined by Fechner in the *Elemente,* was especially appropriate: Although its meaning is derived from the relations "of the material and the mental," it can also be understood as designating the psychology aspiring to be "like physics."

Before Fechner's work, Ernst Weber (1846) systematically experimented with pairwise comparisons of weights and visually presented line segments, but his observations did not lead him beyond an empirical generalization bearing his name. Although called a "law," this generalization played a very different role from, say, Newton's laws of motion, as it was not used

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to derive anything else from it. Johann F. Herbart (1824), on the other hand, constructed an elaborate mathematical theory of "strengths" of mental events (*Vorstellungen*) interacting in one's mind. He did not think it imperative, however, to be able to somehow measure these "strengths" or indeed to be able to identify individual mental events as separate entities. Despite Herbart's titling his principal treatise *Psychology as Science,* it more appropriately falls under the rubric of mathematical metaphorizing.

Unlike Herbart's mental events, Fechner's sensations are identifiable by the stimuli causing them. To reproduce a sensation one simply has to present to the observer the same stimulus under the same conditions (although the conditions may not be entirely controllable if they include the observer's physiological states). Unlike Herbart's nebulous "strengths of mental events," the notion of a sensation magnitude is operationally defined: There are certain empirical and computational procedures that allow one to arrive at numbers representing these magnitudes. The measurements of difference thresholds described in the *Elemente* can be viewed as "merely" elaborate versions of those used before him, but their significance was in something else: They did not interest Fechner for their own sake but rather as a way to inform a mathematical theory of subjective differences and sensation magnitudes. This conjunction of the mathematical and the operational makes the dating of scientific psychology from the *Elemente* amply deserved. Its only historical rival in this respect seems to be Daniel Bernoulli's (1738/1954) admirable analysis of "moral wealth," which can be viewed as having founded the modern theory of decision making, more than a century before Fechner's work.[1]

Not everyone would agree with this characterization of Fechner's work. Many contemporary psychophysicists, in the wake of S. S. Stevens's disparagement of Fechner's theory (Stevens, 1960, 1961), would reduce the principal significance of Fechner's work to the first systematic presentation of the three classical methods of measuring thresholds. It is Fechner's theory that, however, is the exclusive focus of this article. Ever since the publication of the *Elemente,* Fechner was criticized for being conceptually confused when dealing with just-noticeable differences (JNDs) in relation to his psychophysical function, and for using faulty mathematical reasoning in critical derivations (Elsas, 1886; Müller, 1878; Luce & Edwards, 1958). We think this criticism is based on misinterpretation, even if to a large extent it is due to Fechner's own expository and terminological shortcomings. We argue in this article that Fechner's derivations of his logarithmic law are valid, and we discuss in some detail their logical relation to Fechner's methods of measuring thresholds and sensitivity. We then present our extraction from Fechner's theory of what we think to be its most essential and enduring aspects, and we proceed to discuss our understanding of what it is that one can call the main Fechnerian idea, the legacy of Fechner's theory to contemporary psychophysics.

In accomplishing these goals we do not attempt a textual analysis of the *Elemente,* or indeed any of Fechner's other works. Fechner's writing is often less than clear and open to conflicting interpretations. This article is more of a reconstruction than a review or historical analysis: We try to reconstruct the logic of the Fechnerian approach, and we do this using the language acceptable in modern psychophysics rather than Fechner's own words. Our reconstruction, however, is not an alternative reality, a substitution of what ought to have been said for what has been said. We ascribe to Fechner's theory only the positions that are unequivocally contained in Fechner's texts or can be plausibly inferred from them. Thus, it is a fact that the *Elemente* contains two derivations of Fechner's law, one of which is based on presenting a certain principle (which we call the W-principle) as a Cauchy-type functional equation. It is a fact that neither derivation makes use of JNDs; therefore, neither derivation is based on Weber's law or the postulated subjective equality of JNDs (known today as "Fechner's postulate"). It is a fact that the counting of just-noticeable increments leading from one stimulus to another as a procedure for measuring subjective difference between them is understood by Fechner as an approximation only, justified if Weber's fraction is sufficiently small.[2]

The situation is different with our understanding of what constitutes the "main Fechnerian idea": We see it as the idea of summation of differential sensitivity values along an interval of stimulus values, and this choice is determined by our own view of psychophysics and its historical development after Fechner. Even so, our interpretation is consistent with Fechner's views. It is supported, in particular, by Fechner's emphasizing (in the *Elemente* and in his other writings) that the idea of summation of very

small subjective increments would constitute a valid basis for psychophysical measurement even if Weber's law (the term Fechner uses to designate both the law established by Weber and a law of his own, called here the W-principle) were abandoned or replaced by another law, leading to functions other than the logarithmic one.

As a brief biographical note, Fechner was nearly 60 when he published the *Elemente*. Before this event, one might argue, he published nothing of notable scientific value, except possibly for an appendix to his *Zend-Avesta* (1851), in which he described his insight of the logarithmic law. On and off, Fechner continued to be active in psychophysics for almost 30 years after the *Elemente,* having published his last and rather insightful article in 1887, the year of his death at the age of 86. Quite a source of inspiration for aging scientists.

Fechner's Unidimensional World

The most conspicuous feature of Fechner's approach to the relations "of the material and the mental," the feature that has remained ubiquitous in psychophysics up to the present day and in the opinion of many almost defining it, is the unidimensionality of both the material and the mental: The former is represented by unidimensional continua of intensity and extent, the latter by corresponding unidimensional continua of the "sensation magnitudes." Mathematically, both a mental continuum and its "physical correlate" are sets of nonnegative real numbers. The basic relation of the two is simple: Subjective magnitude increases with stimulus magnitude (intensity or extent) beginning with some positive value o of the latter, called the absolute threshold. The value of o is subject to stochastic variability, but we will follow Fechner in acknowledging this but treating it as a constant. We will disregard the issue of "negative," subliminal sensations, which interested Fechner but remained extraneous to his theory. Contrary to the notion that Fechner's theory is critically based on the notion of a JND, the function relating a mental continuum to its physical counterpart is explicitly assumed by Fechner to be continuous (*Elemente,* p. 20 of Vol. 1, and p. 85 of Vol. 2).[3]

However, sensation magnitude need not be taken as a primitive of Fechner's theory. The logic of the latter is more consistent with the view that the notion of sensation magnitude is constructed by means of a more basic concept of difference sensation (*Unterschiedsempfindung*). In Fechner (1887), his last work on psychophysics, Fechner states this explicitly (see p. 9): The notion of sensation magnitude is linked to that of difference sensation through what Fechner calls the "intermediate" concept of sensation difference (*Empfindungsunterschied*), or difference between sensations. The link is established by postulating that the difference sensation for stimuli a and b and the difference between the two corresponding sensations, though logically different notions, are numerically equal to each other: *The sensation of difference between two stimuli is the same as the increment in sensation magnitude from the lesser to the greater of the two stimuli* (cf. *Elemente,* p. 85 of Vol. 2). Stated in modern terms (and replacing the term *difference sensation* with a more modern-sounding *subjective dissimilarity*), for stimuli a and b above or at the threshold value o, their subjective dissimilarity $D(a,b)$ in Fechner's theory has the properties of a unidimensional distance: $D(a,b) = 0$ if and only if $a = b$; $D(a,b) = D(b,a)$; and, whenever $a \le b \le c$,

$$D(a,c) = D(a,b) + D(b,c). \qquad (1)$$

This additivity property is central for Fechner's theory, as he repeatedly states when discussing the notion of measurement (e.g., *Elemente,* pp. 56, 60 of Vol. 1 and chapter 20 in Vol. 2). It is equivalent to assuming that for $o \le a \le b$ the subjective dissimilarity $D(a,b)$ can always be presented as the difference $D(o,b) - D(o,a)$. The quantities $D(o,b)$ and $D(o,a)$ are dissimilarities of the respective stimuli from the absolute threshold o, and it is these quantities that Fechner calls the magnitudes of the sensations caused by, respectively, b and a.[4]

JNDs, Weber's Law, and W-Principle

The notion of a JND (which term we use throughout this article as synonymous to Fechner's *differential threshold*), poses a great, if not the greatest, conceptual problem for Fechner's theory. In chapter 10 of the *Elemente* Fechner subscribes to the notion that if the physiological representations of stimuli a and b are sufficiently close to each other, these stimuli are perceived as precisely the same (Vol. 1, pp. 242–243). At the same time, he thinks of sensation magnitudes of a given kind as forming a continuum and of the psychophysical function as mapping two distinct stimuli, however close, into two distinct sensations.

To justify his "method of right and wrong cases" (now known as the method of constant stimuli), Fechner acknowledges that a stimulus b, however close to a, will be perceived sometimes greater than a and sometimes less than a (*Elemente,* Vol. 1, pp. 77, 247). And it is clear from his use of normal ogives to approximate psychometric functions (discussed later in this article) that, for a fixed stimulus a, the probability $\Pr[a \prec b]$ with which a stimulus b is judged to be greater than a is different for different values of b. It is reasonable therefore to disregard Fechner's belief in true indistinguishability (stated in terms of the relations between physiological and mental processes rather than the latter and stimuli; see *Elemente,* Vol. 1, pp. 248–249) and to ascribe to him the modern psychophysical view according to which a JND is merely an expedient characterization of a distribution of comparative responses to pairs of stimuli, adopted by a convention, such as the difference between two arbitrarily chosen quantiles of a psychometric function or an arbitrarily chosen measure of spread for matching values in the method of adjustments.

Weber's law can be formulated as the statement

$$\frac{a'}{a} = 1 + C^*, \tag{2}$$

where a' denotes a stimulus just noticeably greater than a, and C^* is a positive constant (traditionally referred to as Weber's fraction). The value of a' clearly depends on the method of measurement used and the convention adopted. Thus, with the method of constant stimuli, a' for a given a is defined by

$$\Pr[a \prec a'] = p, \tag{3}$$

where p is some probability value between ½ and 1 (we make here some simplifying assumptions that will be explicated later). In his definition of JND, Fechner sets p equal to 1 with the proviso that $\Pr[a \prec b] < 1$, for any $b < a'$ (*Elemente,* Vol. 1, p. 128). But his analysis of pairwise comparisons of weights (*Elemente,* Vol. 1, pp. 182–201) shows that he thought Weber's law applied to any value of $p > ½$. The value of C^* then depends on one's choice of p: The closer the latter to ½, the smaller the C^*.

If the method used is that of the "average error" (the method of adjustment, or matching), then a' can be defined as

$$a' = a + k\sigma(a), \tag{4}$$

where $\sigma(a)$ is some measure of spread (say, standard deviation) of stimuli judged to match a, and k is an arbitrary positive constant. There can be no justification for preferring one value of k to another, and although one's choice of k does not affect the validity of Weber's law, it affects the value of C^*: The smaller the k, the smaller the C^*.

That C^* depends on p in the method of constant stimuli and on k in the method of adjustment and that in both cases C^* can be made arbitrarily small will be seen to be important for operational aspects of Fechner's theory. Even with the "method of JNDs" (known today as the method of limits), where a' is defined as a measure of central tendency $\mu(a)$ of the distribution of stimuli judged to be just noticeably greater than a, one can argue that the value of C^* can be made arbitrarily small by using an arbitrarily small positive k in the modified definition

$$a' = a + k(\mu(a) - a). \tag{5}$$

Fechner knew that this method can be trusted less than the other two because $\mu(a)$ in it is greatly affected by "subjectivity" (*Elemente,* Vol. 1, p. 75), that is, observers' decision-making criteria.

We see that the definition of a' in Weber's law does not relate to the notion of subjective dissimilarity $D(a,b)$ in any direct way. Response probabilities, standard deviations of matches, and the means of stopping points in sequences of stimuli are all objective characteristics of observable response distributions. However, it is widely believed, and parts of the *Elemente* (e.g., pp. 59, 68 of Vol. 1 and pp. 58, 428 of Vol. 2) may indeed be interpreted as suggesting this, that Fechner assumed that any two stimuli separated by a JND (at least for some methods and conventions used to define it) have a fixed degree of subjective dissimilarity:

$$D(a,a') = C, \tag{6}$$

where C is some positive constant (which may be different for different physical continua, say, intensities of tones of different frequencies). This statement is sometimes called "Fechner's postulate." With this postulate accepted, Weber's law can be augmented into the statement

Weber's Law + Fechner's Postulate. With an appropriate definition of JND, the ratio a'/a of

any two stimuli separated by one JND is constant, and so is the dissimilarity between these stimuli, $D(a,a')$.

Later in this article (where we discuss Elsas's criticism) we will see that this formulation creates difficulties for Fechner's theory.

For now, however, observe the following consequence of this formulation: Denoting by $a^{(n)'}$ the stimulus separated from the smaller stimulus a by n JNDs ($n = 1, 2, \ldots$), we have

$$\frac{a^{(n)'}}{a} = \frac{a^{(n)'}}{a^{(n-1)'}} \times \ldots \times \frac{a'}{a} = (1 + C^*)^n$$

and

$$D(a,a^{(n)'}) = D(a,a') + \ldots + D(a^{(n-1)'},a^{(n)'}) = nC.$$

In other words, equal ratios of stimulus magnitudes give rise to equal dissimilarities, provided the stimuli are separated by an integer number of JNDs. Fechner does not use this reasoning explicitly, but he formulates a statement that he calls "Weber's law," which can be viewed as the previous formulation but without mentioning JNDs:

> **W-principle.** The subjective dissimilarity between stimuli with physical magnitudes a and b (provided $o \le a \le b$, where o is absolute threshold) is determined by the ratio of these magnitudes, b/a.

We will call this statement the W-principle to allude to the fact that in the *Elemente* Fechner called it "Weber's law" (chapter 9 in Vol. 1) and at the same time to distinguish it from Weber's law conventionally understood as referring to (2). Stated in symbols, this principle says that $D(a,b)$ is some function of b/a. Fechner's own numerous formulations are almost equally precise (e.g., the concise formulation "sensation differences or sensation increments remain the same as long as ratios of stimuli remain the same," on p. 134 of Vol. 1 of the *Elemente*).

Note that Weber's law (2), "Fechner's postulate" (6), and the W-principle are logically independent, in the sense that each can hold true with the other two being false. If the W-principle is postulated, however, then "Fechner's postulate" and Weber's law logically imply each other: Both must be true or both false, for any given way of measuring JNDs. The conjunction of Weber's law with "Fechner's postulate" does not imply the W-principle, except for pairs of stimuli separated by an integer number of JNDs (as explained later, one needs "infinitesimal" versions of Weber's law and "Fechner's postulate" to be able to derive the W-principle from them in full generality). We cannot be certain that Fechner was clear in his mind about all these logical relations.

With this preamble, let us consider how Fechner derives his celebrated law.

Fechner's Forgotten Derivation of Fechner's Law

The controversial derivation believed to involve Weber's law, the postulated subjective constancy of JNDs, and a certain differential equation is well known (to English-language readers, primarily from Boring's 1950 account). We will deal with it in the next section. It is less known (perhaps even entirely forgotten in the post-Fechner psychology) that chapter 17 of the *Elemente* contains a derivation of Fechner's law from the W-principle formulated in chapter 9, both the formulation and the derivation making no use of JNDs and involving no differentiability assumptions.

The W-principle says that if $o \le a \le b$,

$$D(a,b) = F(b/a), \tag{7}$$

where F is some function.[5] This statement is equivalent to Fechner's logarithmic law. Indeed, the statements (7) and (1) combine into the following: Whenever $o \le a \le b \le c$,

$$F(c/b) + F(b/a) = F(c/a). \tag{8}$$

It is not difficult to show, by transforming this equation into what is known as the Cauchy functional equation on positive reals (see Aczél, 1987), that the only regular (in particular, nonnegative) function F that satisfies it is

$$F(x) = K \log x, \quad x \ge 1, \tag{9}$$

where K is some positive constant.[6] It follows that (7) and (1) can be satisfied if and only if, for any $o \le a \le b$,

$$D(a,b) = K \log \frac{b}{a}. \tag{10}$$

This is Fechner's "difference formula" (*Unterschiedsformel*). In particular, the sensation magnitude $D(o,a)$ for a stimulus $a \le o$ is computed as

$$D(o,a) = K \log \frac{a}{o}, \tag{11}$$

which is the traditional formulation of Fechner's law (called by Fechner the "measurement formula," *Massformel*).

The derivation of this law presented in chapter 17 of the *Elemente* (pp. 33–39 of Vol. 2) is essentially the same as the modernized account just given. Fechner correctly derived (8) and recognized it as a functional equation of the type treated by Augustin-Louis Cauchy (1821/2009). From this work, by then only 40 years old, Fechner knew that (9) is the only solution for F in (8). With some caveats related to unexplicated assumptions and the domains of the functions involved, Fechner's reasoning in this derivation is sound and rigorous.

Fechner's use of the term *Weber's law* to designate the W-principle, (7) (i.e., an equivalent to what we now call Fechner's law), could have been merely a token of Fechner's respect for Weber. But it created a conceptual confusion that began dogging him during his lifetime and has lasted until the present time. As an example, G. E. Müller (1878) essentially replicated Fechner's derivation of the logarithmic law by reducing it to a Cauchy-type functional equation but claimed superiority because Fechner's derivation, in his words, "is making use of the so-called fact of a stimulus threshold, without which, as Fechner argues erroneously, the logarithmic relation between sensation intensity and stimulus intensity cannot be derived" (p. 228). Müller was right to think of "stimulus threshold" as a theoretically unnecessary concept for Fechner's derivation of the logarithmic law, but he was wrong in assuming that Fechner made use of this concept.

The Well-Known (but Misunderstood) Derivation

The juxtaposition of "Fechner's postulate" and Weber's law leads us to the second, better-known, and often criticized derivation of Fechner's law (see Elsas, 1886; Luce & Edwards, 1958; Makarov, 1959; Stevens, 1960). It is given in chapter 16 of the *Elemente* and is usually described by Fechner's critics as follows. Since C in Fechner's postulate (6) is a constant, and since C^* in Weber's law (2) is another constant, one can write

$$D(a,a') = D(o,a') - D(o,a) = \frac{C}{C^*} \cdot \frac{a'-a}{a}. \qquad (12)$$

Then, the critical account goes on, Fechner invokes an "expedient principle" (*Hülfsprinzip*) according to which $a' - a$ in this formula can be replaced with an infinitesimal change da. Assuming the differentiability of the sensation magnitude $D(o,a)$ with respect to the stimulus magnitude a, the difference $D(o,a') - D(o,a)$ becomes an infinitesimal increment $dD(o,a)$. The statement (12) from a trivial relation between two constants turns into an informative differential equation,

$$dD(o,a) = \frac{C}{C^*}\frac{da}{a}, \qquad (13)$$

whose solution is, for any $o \le a \le b$,

$$D(o,b) - D(o,a) = D(a,b) = \frac{C}{C^*}\log\frac{b}{a}. \qquad (14)$$

In this way one gets Fechner's "difference formula" (14) with $K = C/C^*$.

Of course, this derivation is mathematically flawed. Moreover, as a physicist Elsas (1886) pointed out, there is an inconsistency between (12) and (14). A simple way of demonstrating it is this: Since $a' = (1 + C^*)a$ according to (2), the subjective dissimilarity $D(a,a')$, according to (14), should equal

$$D(a,a') = \frac{C}{C^*}\log\frac{a'}{a} = \frac{C}{C^*}\log(1 + C^*). \qquad (15)$$

Because this quantity is different from C, this formula contradicts "Fechner's postulate" (6). In his 1887 article, shortly before his death, Fechner responded to this criticism by saying that Weber's law (2) and the differential equation (13) can be related to each other only if C^* is sufficiently small (Fechner, 1887, p. 166). We understand this rejoinder to support the following understanding.

The factual derivation of Fechner's law in chapter 16 of the *Elemente* does not make use of Weber's law at all. It makes use of the W-principle (which, we keep in mind, Fechner called "Weber's law"). If (7) holds and the function $F(x)$ is assumed to be differentiable at least at $x = 1$ (an innocuous assumption for a 19th-century scientist), then

$$D(a,a + da) = dD(o,a) = K\frac{da}{a}, \qquad (16)$$

where $K = F'(1)$. This is Fechner's "basic formula" (*Fundamentalformel*), whose solution is the logarithmic function (10). If, in addition to the W-principle, Weber's law (2) happens to hold too (or, equivalently, "Fechner's postulate" (6) happens to hold too), then

$$D(a,a') = K\log\frac{a'}{a} = K\log(1 + C^*) = C. \qquad (17)$$

Now, if C^* is very small, then $\log(1 + C^*) \approx C^*$, whence the interpretation of K as C/C^* in (14) and

(13) becomes *approximately* correct, and no contradiction ensues. In particular, the expression in (15) is *approximately* equal to C.

It was not too risky for Fechner to assume that C^* is small. Recall that if one uses the method of constant stimuli, C^* can be made arbitrarily small by choosing p in the defining formula (3) sufficiently close to ½ (assuming, as Fechner apparently did, that Weber's law holds for any p). If one uses the adjustment method or the method of limits, C^* can also be made arbitrarily small by choosing arbitrarily small value of k in the defining formulas (4) and (5). It is not clear, however, whether Fechner himself was aware of this "small-by-construction" consideration.

It seems to us that Fechner's "expedient principle" (chapters 15 and 16 of Vol. 2), the main target of his critics' derision (Luce & Edwards, 1958), is merely an inept way of describing the transition from (7) to the differential equation (16): The difference sensation $D(a,b)$ is some function of $(b-a)/a$, Fechner says,

$$D(a,b) = F(b/a) = G\left[\frac{b-a}{a}\right],$$

and the "expedient principle" ensures that as b tends to a, making $(b-a)/a$ infinitesimally small, the relationship between $D(a,b)$ and $(b-a)/a$ tends to a simple proportionality,[7]

$$D(a,a+\mathrm{d}a) = G\left[\frac{\mathrm{d}a}{a}\right] = G'(0)\frac{\mathrm{d}a}{a}.$$

This is trivially correct, and this would explain why Fechner says that this otherwise bizarre principle is a priori valid. Our interpretation is also corroborated by the fact that in his 1877 treatise (p. 10) Fechner states that his "basic formula" can be presented *equivalently* as (16) or as (7).

Link to Threshold Measurements

The question arises: If Weber's law and JNDs do not play any role in Fechner's derivations of his law, what is the role of Fechner's methods of measuring JNDs? We mentioned at the beginning of this article that the pioneering status of the *Elemente* is in the conjunction of the mathematical and the operational. Where does the latter enter his theory?

We know that if both the W-principle (from which alone Fechner's law is derived) and Weber's law are satisfied, then so is "Fechner's postulate" in the form (17), and Fechner's difference formula (10) can be written as

$$D(a,b) = \frac{C}{\log(1+C^*)}\log\frac{b}{a}, \tag{18}$$

for $o \le a \le b$. Let us count the approximate number of JNDs that fit between a and b. Weber's law implies that, starting from a, one can form the geometric progression $a, a(1+C^*), a(1+C^*)^2, \ldots$, until, for some n,

$$a(1+C^*)^n \le b < a(1+C^*)^{n+1} \tag{19}$$

For any given a and b, the smaller the value of C^* the smaller the interval containing b. By simple algebra,

$$n \le \frac{1}{\log(1+C^*)}\log\frac{b}{a} < n+1. \tag{20}$$

But we can also estimate the number of JNDs fitting between a and b by taking the subjective dissimilarity $D(a,b)$ in (18) and dividing it by the fixed value C of the dissimilarities corresponding to JNDs:

$$\frac{D(a,b)}{C} = \frac{1}{\log(1+C^*)}\log\frac{b}{a}. \tag{21}$$

Because the logarithmic expression in (21) coincides with that in (20), it falls between the same two n and $n+1$. That is, *the number of JNDs fitting between two given stimuli approximately equals the subjective dissimilarity between the two stimuli measured in units of the subjective value of the JNDs*. This links Fechner's theory with empirically measured JNDs and empirically measured Weber's fraction C^*. This is essentially what the traditional interpretation of Fechner's theory boils down to, and we see that this interpretation can be upheld without finding faults with Fechner's reasoning and use of mathematics.

Beyond Weber's Law and W-Principle

Fechner repeatedly says in the *Elemente* (e.g., on pp. 65–66 of Vol. 1 and p. 2 of Vol. 2) that the applicability of his approach to (or his "principle of") measuring sensation magnitudes is not contingent on the validity of "Weber's law," and it is never quite clear whether he means the W-principle or Weber's law. As stated earlier, it is possible he was not clear in his mind about the difference. He must mean Weber's law when he says that it is empirically known to be violated outside a middle range of stimulus magnitudes (p. 67 of Vol. 1, and pp. 195, 336, and 429 of Vol. 2). But in other places the context suggests that he means the W-principle (pp. 9–10 and 34–36 of Vol. 2). Either understanding raises questions. With Weber's law abandoned (and the W-principle intact),

Fechner's theory seems to lose its operational link to threshold measurements. Why then should one postulate the W-principle? Why not to assume instead that $D(a,b)$ is a continuous function of stimulus difference, for instance? Or the difference of square roots of stimulus values? On the other hand, if the W-principle is no longer a postulate, then what is this "principle" of measurement mentioned by Fechner, one general enough to transcend the W-principle and specific enough to allow one to base derivations of the psychophysical function on it?

We think that the key to the answer lies in the general notion of additive dissimilarity introduced earlier. If we could somehow figure out a measure of subjective dissimilarity between any given stimulus and a "slightly greater" stimulus, and if, for any two stimuli $a < b$, a sequence

$$a < a' < a'' < \ldots < a^{(n)'} \approx b \tag{22}$$

could be constructed whose each member is "slightly greater" than the previous one, then $D(a,b)$ could be approximately computed as

$$D(a,b) \approx D(a,a') + \ldots + D\left(a^{(n-1)'},a^{(n)'}\right). \tag{23}$$

Fechner speaks of this summation "principle" at length in the *Elemente* (p. 60 of Vol. 1, and Chapters 20 and 31 of Vol. 2). One possibility now is to accept "Fechner's postulate" (6) and think of the "slightly greater" in the construction of the sequence (22) as meaning "just-noticeably greater." If the subjective dissimilarity between any two successive stimuli in such a sequence has a fixed value C, then

$$D(a,b) \approx nC. \tag{24}$$

Elemente does not seem to suggest this as a general solution, however. Moreover, this is hardly an acceptable solution for the conceptual problem of measuring dissimilarities. Approximate equality can exist only between two quantities that have precise values. What is the definition of the precise value of $D(a,b)$, even if we are satisfied with estimating it approximately? What is the dissimilarity between two stimuli separated by less than one JND, however defined and however small? Fechner is aware of this problem: In his discussions of the summation (23) he emphasizes that the closer to each other the successive stimuli can be taken, the more precisely $D(a,b)$ can be measured for arbitrary pairs (a,b). Ideally therefore, Fechner says (*Elemente*, p. 60 of Vol. 1 and p. 65 of Vol. 2), the intervals between the successive stimuli in (23) should be infinitesimally small.

Our interpretation of this is that Fechner in effect proposes to present $D(a,b)$ for $o \le a \le b$ as an integral rather than a finite sum:

$$D(a,b) = \int_a^b H(x)\mathrm{d}x\,, \tag{25}$$

where

$$H(x) = \frac{D(x,x+\mathrm{d}x)}{\mathrm{d}x}. \tag{26}$$

One can consider (26) the generalized "basic formula," with (25) being the correspondingly generalized "difference formula." Fechner's theory discussed in the previous sections can be viewed as based on the assumption that

$$H(x) = K / x, \tag{27}$$

an "infinitesimal version" of Weber's law. In his 1877 monograph (chapter 4), however, Fechner discusses a variety of possible alternatives to this function, including (p. 21)

$$H(x) = K / x^{\alpha}, \tag{28}$$

with $0 < \alpha < 1$. This yields

$$D(a,b) = K^*\left(b^{1-\alpha} - a^{1-\alpha}\right), \tag{29}$$

a variant (proposed by Plateau, 1872, and Brentano, 1874) of what later became known as Stevens's psychophysical function (Stevens, 1975).

The question now is, What is a principled way of choosing a particular form of the function $H(x)$? Can one reconstruct it from empirical observations? An answer can be found in Fechner's analysis of his "method of right and wrong cases." Let us agree to always present stimulus pairs (a,b) so that $a \le b$, and let observers' responses be classified as "$a \prec b$" (correct response) and "$a \succ b$" (incorrect response). If the response "equal" is allowed, it is supposed to be split between these two categories according to some rule. To simplify the discussion, let us assume that $\Pr[a \prec b]$ does not depend on any stimulus characteristics but their magnitudes a and b. For instance, it does not matter which of two tones, a or b, is presented first; or which of two lights is on the left and which on the right; or which of two weights is the standard held fixed within a block of trials, and which

is the varying comparison weight.[8] So the simplification consists in presenting $\Pr[a \prec b]$ as some function $\gamma(a,b)$, defined on the set of all pairs with $a \leq b$. It is clear then that $\Pr[a \succ b] = 1 - \gamma(a,b)$, whence $\gamma(a,a) = \frac{1}{2}$. Now, Fechner's assumption about $\gamma(a,b)$ can be presented as

$$\gamma(a,b) = \Phi\big[h(a)(b-a)\big], \tag{30}$$

where Φ is the standard normal integral and $h(a)$ some positive number (*Elemente,* pp. 102 and 107 of Vol. 1). One can interpret the right-hand expression as the probability with which a random variable **A** normally distributed with the mean a and standard deviation $1/h(a)$ falls below the value b. Fechner thinks of $\mathbf{A} - a$ as some kind of a measurement error, whence it becomes natural to interpret $h(a)$ as the "precision" of a measuring device (the term used by Carl Friedrich Gauss [1809/2004] in his introduction of normal distribution as "the law of errors"). Using this analogy, Fechner takes $h(a)$ as a measure of sensitivity at stimulus a. He thinks that empirical evidence corroborates the inverse proportionality $h(a) = K/a$ and takes this as a way of establishing Weber's law (*Elemente,* pp. 182–201 of Vol. 1). Note that the notion of sensitivity and Weber's law here are not defined in terms of JNDs.

We propose that the sensitivity function thus derived, modulo a scaling constant, can be taken as the function H in (26): For all $x \geq o$,

$$H(x) = Kh(x). \tag{31}$$

We propose that this reconstruction of the sensitivity function from psychometric functions can be taken as an illustration of Fechner's general "principle," one that transcends Weber's law and the W-principle. The "principle" itself can be stated in the form of the following instructions:

1. For each suprathreshold stimulus x, determine empirically (e.g., by means of one of Fechner's methods) a quantity $H(x)$ that can be interpreted as a measure of discriminability of x from its neighboring stimuli.
2. Call $H(x)$ the sensitivity at x and identify it with $H(x)$ in the "basic formula" (26).
3. Integrate $H(x)$ from a to b in accordance with the "difference formula" (25) to obtain the value of subjective dissimilarity $D(a,b)$ between stimuli a and b.

Once this approach is implemented using the method of constant stimuli, it is not difficult to extend it to the other two methods. In the adjustment method each stimulus a corresponds to a measure of spread $\sigma(a)$ of the values of stimuli b judged to match a in different trials; the sensitivity measure here can be chosen as $K/\sigma(a)$. In the method of limits the sensitivity measure can be chosen as a quantity inversely proportional to the mean increment in comparison series, $K/(\mu(a) - a)$. The latter two definitions reflect Fechner's considering the reciprocal of any measure of JND as the corresponding measure of sensitivity (we remind the reader that we use the term *JND* as synonymous to *differential threshold*). This definition of sensitivity is well known and is considered classic. However, the procedure with fitting a normal ogive to response probabilities and gauging how fast it grows shows that Fechner could think of alternative ways of measuring sensitivity, circumventing an explicit use of the notion of a JND. In fact, the introduction of the probabilities of comparative responses makes the notion of a JND theoretically unnecessary and provides a bridge from Fechner's theory to modern psychophysics.

Post-Fechner Developments of the Fechnerian Idea

Presentations of the modern version of the Fechnerian idea can be found in Dzhafarov and Colonius (1999, 2001) and Dzhafarov (2001). We describe here a simplified version, continuing to treat $\Pr[a \prec b]$ as a function $\gamma(a,b)$, defined on the set of all pairs with $a \leq b$.

Fechner's reliance on the normal ogive (30) is certainly an overcommitment to a particular model (later dubbed in psychophysics the "phi–gamma hypothesis"). A safer approach would be applicable to psychometric functions of any shape. A key to such a general approach can be found in the fact that the sensitivity measure $h(a)$ in (30) is proportional to the slope of the tangent drawn to the psychometric function $b \mapsto \gamma(a,b)$ at its median (under our simplifying assumptions, at the point $b = a$, where $\gamma = \frac{1}{2}$):

$$\frac{\partial \Phi\big[h(a)(b-a)\big]}{\partial b}\Big|_{b=a+} \propto h(a). \tag{32}$$

One gets the same proportionality relation if one replaces Fechner's ogive (30) with one of many modi-

fications, which some researchers (e.g., Thurstone, 1928) would find superior to the phi–gamma hypothesis, such as

$$\gamma(a,b) = \Phi\left(\frac{b-a}{\sqrt{\sigma_a^2 + \sigma_b^2}}\right). \quad (33)$$

Intuitively, the steeper the slope of the tangent at the median, the more discriminable a is from its immediate neighbors to the right.

These observations suggest the following generalization: Use (25) to compute $D(a,b)$ by putting

$$H(a) = C\frac{\partial\gamma(a,b)}{\partial b}\Big|_{b=a+}, \quad (34)$$

where C is some positive constant characterizing a given stimulus continuum. Of course, one has to assume that the right-hand derivative in (34) always exists and that it is positive and integrable in a. Once we have accepted (34), the use of formula (25) is equivalent to assuming that

$$\lim_{b\to a}\frac{D(a,b)}{\gamma(a,b) - 1/2} = C. \quad (35)$$

This can be viewed as an infinitesimal version of "Fechner's postulate," provided that the notion of a and a' separated by a JND is operationalized as in (3). This idea, according to Krantz (1971), has been proposed by M. Frank Norman (currently a professor emeritus at the University of Pennsylvania).

By 1960s, the prevailing opinion among psychophysicists was that Fechner's approach was mathematically flawed, and it was proposed in Luce and Edwards (1958) and Luce and Galanter (1963) that it should be reformulated as the following "Fechner's problem": Given a psychometric function $\gamma(a,b)$, find an additive distance function $D(a,b)$ such that

$$\gamma(a,b) = G(D(a,b)) = G(D(o,b) - D(o,a)), \quad (36)$$

where G is some increasing function (possibly with additional regularity constraints). This problem was extensively investigated by Falmagne (1971, 1985). However, Pfanzagl (1962) noticed that if "Fechner's problem" has a solution with a function G differentiable at zero, then $D(a,b)$ should satisfy (25), with $H(a)$ satisfying (34). Krantz (1971) essentially replicated this observation. The reverse clearly is not true: (25) and (34), or equivalently (34) and (35), can hold without "Fechner's problem" being solvable. This shows that what we call the Fechnerian idea gives rise to a more general and flexible scaling procedure than "Fechner's problem."

Long before the formulation of "Fechner's problem," Hermann von Helmholtz and Erwin Schrödinger (better known for his contribution to quantum mechanics) intuitively grasped the Fechnerian idea and saw in it a potential for breaking away from Fechner's unidimensional world. Formulated in terms of the present article, these authors proposed to treat the square of the function $H(x)$ in (25), which they took to have the Weber-law form K/x, as a metric tensor of unidimensional Riemannian geometry. This suggested to them that for stimuli represented by vectors of real numbers one could compute Riemannian distances between them by using metric tensors generalizing the function K/x. Helmholtz (1891) and Schrödinger (1920) applied two different generalizations to the three-dimensional color space (see Dzhafarov & Colonius, 1999, for details). Riemannian geometry based on differential thresholds measured along arbitrary directions in color space remains the main mathematical language of color science (Robertson, 1978; Wyszecki & Stiles, 1982). Color science therefore has always been very Fechnerian in flavor, essentially untouched by the "power versus logarithm" debate that for more than half a century preoccupied psychophysicists. The potential of the Fechnerian idea to transcend unidimensionality is not shared by Stevens's approach.

A more direct extension of the Fechnerian idea than in Helmholtz and Schrödinger was recently proposed in Dzhafarov and Colonius (1999, 2001) and Dzhafarov (2002a, 2002b, 2002c, 2004). If stimuli are represented by points in a multidimensional Euclidean space, then any point $\mathbf{a}$ and any direction of change $\mathbf{u}$ attached to it can be associated with a Finslerian metric function $H(\mathbf{a},\mathbf{u})$, a multidimensional analog of the function $H(a)$. The computation of $H(\mathbf{a},\mathbf{u})$ generally cannot be based on the "probability of greater/less" psychometric functions, because the responses "greater" and "less" require the existence of a semantically linearly ordered property, such as saturation, brightness, or beauty. It was proposed therefore to use the function

$$\psi(\mathbf{a},\mathbf{b}) = \Pr[\mathbf{a} \text{ and } \mathbf{b} \text{ are judged to be different}],$$

whose estimates can be empirically obtained by presenting to observers pairs of stimuli and asking them

to judge them as "different" or "the same," overall or in a specified respect (e.g., shape, color, beauty, or brightness). Any two stimuli **a** and **b** can be connected by a continuously differentiable path, and the metric function $H(\mathbf{a},\mathbf{u})$ can be integrated along this path, a multidimensional analog of (25). This yields what we call the "psychometric length" of the path, and the distance between **a** and **b** can be defined as the greatest lower boundary for such psychometric lengths obtainable across all possible smooth paths connecting **a** to **b**.

In the course of the development of this "Finslerian" extension of Fechner's idea it has become clear it can be further extended to stimulus spaces more general than multidimensional manifolds (Dzhafarov & Colonius, 2005a, 2005b). It is even possible to construct such an extension for discrete, including finite, stimulus spaces (Dzhafarov & Colonius, 2006a), allowing one to use the generalized Fechnerian scaling as a data analytic technique rivaling or complementing, depending on one's preference, the widely used multidimensional scaling (Dzhafarov, 2010a). The idea of cumulating dissimilarities in a discrete space turned out to be the foundation for extending the Fechnerian idea to stimulus spaces of a completely arbitrary nature (e.g., the space of human faces or the space of dynamic scenes). This "ultimate" extension is described in Dzhafarov and Colonius (2007) and Dzhafarov (2008a, 2008b, 2010b).

Summary

The two derivations of Fechner's law we find in the *Elemente* are merely two ways of solving the same functional equation, (7). Both derivations are mathematically valid, and the only difference between them is in that the better-known one assumes the differentiability of F at a particular point. Neither derivation uses the notion of a JND, so neither is based on Weber's law. However, if Weber's law is empirically established (with the aid of Fechner's classic methods and appropriate conventions), then the functional equation in question ensures that the dissimilarity between any two just-noticeably different stimuli is a fixed quantity C. Then the dissimilarity between any two stimuli, if measured in C-units,[9] approximately coincides with the number of JNDs that fit between these two stimuli: The smaller Weber's fraction C^*, the better the approximation (and we know that the value of C^* can be made arbitrarily small). The main weakness of Fechner's derivations of his law is expository: By using the same term, "Weber's law," for both the W-principle and Weber's law, Fechner has created a lasting confusion.

The W-principle and Fechner's derivations of the logarithmic law do not constitute the essence of his approach. We see this essence in the idea of additive cumulation of local sensitivity values $H(x)$, as in (25) and (26). The function $H(x)$ is established by using Fechner's methods of determining a measure of discriminability of stimuli from their neighboring stimuli. So understood, the idea is immune to many criticisms, mathematical and empirical, leveled against Fechner's theory during the 150 years since the publication of his principal work. Moreover, unlike the logarithmic law (or, indeed, any other form of a function relating "sensation magnitude" to "stimulus magnitude"), this idea lends itself to generalizations of Fechnerian measurements to stimuli that are not judged in terms of semantically unidimensional properties (e.g., brightness, loudness, or length) and to stimuli whose physical description is not unidimensional, consisting, for example, of vectors of real numbers, of numerical functions, and even of stimulus names alone.

NOTES

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1. Bernoulli postulates that one's perception of a small increment dx in one's wealth x is inversely proportional to x, sets up the differential equation $dy = Kdx/x$ (where y stands for the subjective value of x, and K is a positive constant) subject to the condition $y = 0$ for some small value x_0 of x, and arrives at $y = K\log(x/x_0)$, for all $x \geq x_0$. He uses this formula to account for the empirical fact that there is an upper limit for the amount of money one is willing to pay for one's participation in the so-called Saint Petersburg game, whereas the expected gain in this game is infinite (the game consists of tossing a coin repeatedly and winning 2^n rubles if a head occurs in the nth toss for the first time). The operational aspect of Bernoulli's approach is obvious: One can test his theory of subjective utility by experimentally determining costs of

appropriately designed gambles. Fechner reviews Bernoulli's work in chapter 47 of the *Elemente,* among other precursors of Fechner's psychophysical function.

2. In Dzhafarov and Colonius (1999) our position was that the traditional view that Fechner's derivation of his law is mathematically flawed was only an interpretation rather than a fact. However, we also wrote that "[Fechner's] writings are too voluminous and complex to dismiss the traditional interpretation of his theory with certainty" (p. 241). Our position at present is different: We now reluctantly admit that the interpretation in question is simply untenable in view of what Fechner factually wrote, his verbosity notwithstanding. We say "reluctantly" because we do not want to be seen as joining those who, in their reverence for the grand and famous, tend to ascribe to them all kinds of views and developments currently deemed clever.

3. All references to the *Elemente* made in this article are to the original German publication. Only the first volume of the book has been translated into English (Fechner, 1860/1966).

4. Titchener (1905) criticized Fechner for attributing magnitudes to sensations per se rather than defining them through dissimilarities. "The only thing that we can measure is the distance between two sensations or sense points" (p. 25). But Titchener acknowledged that "Fechner had an inkling of the truth; he knew that sense-distances are magnitudes, and every now and then he seems to look upon the single sensation as merely the limiting point of a distance" (pp. 26–27). Whatever the evolution of Fechner's factual thinking, nothing in the *Elemente* seems to contradict this "now and then" position, unequivocally accepted in Fechner (1887).

5. Hye Joo Han pointed out to us that Fechner did not need to stipulate any regularity conditions here because F is obviously nonnegative, and this alone is sufficient to derive the logarithmic law.

6. The transformation consists in putting $\exp(x) = c/b$, $\exp(y) = b/a$, and $F(\exp(\cdot)) \equiv G(\cdot)$. The equation then becomes $G(x) + G(y) = G(x + y)$, with x and y arbitrary positive numbers. Cauchy's 1821 solution was predicated on the assumption that F (equivalently, G) is continuous. It is known now (Aczél, 1987) that this assumption can be replaced with many other regularity assumptions, including monotonicity and nonnegativity.

7. That this is indeed what Fechner means can be extracted from the more general (and, unfortunately, entirely verbal) descriptions of the "expedient principle" in chapter 15 of the *Elemente*. Thus, on. p. 7 of Vol. 2 Fechner says that the principle in question ensures that "increments of two interdependent continuous quantities . . . are essentially (*merklich*) proportional as long as they remain very small, whatever the nature of their interdependence and however much the interdependence in the large may deviate from the law of proportionality." We take this to be a convoluted way of saying that for a differentiable function $y = f(x)$, $dy = f'(x_0)dx$ at any $x = x_0$ (hence $\Delta y \approx f'(x_0)\Delta x$ for sufficiently small Δx). See also the mentions of the principle on pp. 11 and 103 of Vol. 2 of the *Elemente*.

8. This simplification is untenable in the general theory of pairwise comparisons: Stimuli must be characterized not only by their values but also but what is called in Dzhafarov (2002d) *observation area*: being first or second, left or right, and so on. Without explicitly encoding stimuli by their observation areas, one's analysis is likely to run into logical difficulties. For detailed accounts see Dzhafarov (2006), Dzhafarov and Colonius (2006b), and Dzhafarov and Dzhafarov (2010).

9. For any given stimulus continuum, C can be set equal to unity. As mentioned earlier, however, the value of C should be treated as modality specific. This consideration is important in evaluating the attempts to empirically invalidate Fechner's theory based on the number of JNDs fitting between isosensitivity curves (Riesz, 1933): This number need not be the same for different sound frequencies if the latter are treated as distinct stimulus continua (see Dzhafarov, 2001, for details; for another way to reanalyze Riesz's results, see Dzhafarov & Colonius, 1999).

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